Partial proof terms in the study of idealized proof search

José Espírito Santo[1] Ana Catarina Sousa[1]

[1]Centre of Mathematics, University of Minho

CICM 2024
Montréal, 8 August 2024

Overview

Overview

- Theoretical study of proof search
- The use of proof terms in proof search

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens
- Generalization of the proof term representation of proofs

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens
- Generalization of the proof term representation of proofs
- Partial proof terms have occurrences of formal sequents

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens
- Generalization of the proof term representation of proofs
- Partial proof terms have occurrences of formal sequents
- Proof search as rewriting of partial proof terms

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens
- Generalization of the proof term representation of proofs
- Partial proof terms have occurrences of formal sequents
- Proof search as rewriting of partial proof terms
- Test the approach with different proof formats: sequent calculus and natural deduction

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens
- Generalization of the proof term representation of proofs
- Partial proof terms have occurrences of formal sequents
- Proof search as rewriting of partial proof terms
- Test the approach with different proof formats: sequent calculus and natural deduction
- Proof search in sequent calculus: focusing
- Proof search in natural deduction: intercalation

Overview

- Theoretical study of proof search
- The use of proof terms in proof search
- Certain level of abstraction: idealized proof search
- Partial proofs are first-class citizens
- Generalization of the proof term representation of proofs
- Partial proof terms have occurrences of formal sequents
- Proof search as rewriting of partial proof terms
- Test the approach with different proof formats: sequent calculus and natural deduction
- Proof search in sequent calculus: focusing
- Proof search in natural deduction: intercalation
- A theorem in IPS: focusing and intercalation are isomorphic

LJT

LJT

Sequents $\sigma ::= \Gamma \vdash A \quad \tau ::= \Gamma | A \vdash p$

Inference rules

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B}\ RI \qquad \frac{\Gamma, x : A | A \vdash p}{\Gamma, x : A \vdash p}\ CTR$$

$$\frac{\Gamma \vdash A \quad \Gamma | B \vdash p}{\Gamma | A \supset B \vdash p}\ LI \qquad \frac{}{\Gamma | p \vdash p}\ AX$$

LJT with proof terms

Proof terms and proof lists

$$\begin{array}{rcl} t, u & ::= & \lambda x^A.t \mid x\hat{}l \\ l & ::= & \star \mid u :: l \end{array}$$

Sequents $\Gamma \vdash t : A \quad \Gamma | A \vdash l : p$

Inference rules

$$\frac{}{\Gamma | p \vdash \star : p}\, AX \qquad \frac{\Gamma, x : A | A \vdash l : p}{\Gamma, x : A \vdash x\hat{}l : p}\, CTR$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \supset B}\, RI \qquad \frac{\Gamma \vdash u : A \quad \Gamma | B \vdash l : p}{\Gamma | A \supset B \vdash u :: l : p}\, LI$$

LJT: proof search (focusing)

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B}\ RI \qquad \frac{\Gamma, x : A | A \vdash p}{\Gamma, x : A \vdash p}\ CTR$$

$$\frac{\Gamma \vdash A \quad \Gamma | B \vdash p}{\Gamma | A \supset B \vdash p}\ LI \qquad \frac{}{\Gamma | p \vdash p}\ AX$$

LJT: proof search (focusing)

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B}\, RI \qquad \frac{\Gamma, x : A | A \vdash p}{\Gamma, x : A \vdash p}\, CTR$$

$$\frac{\Gamma \vdash A \quad \Gamma | B \vdash p}{\Gamma | A \supset B \vdash p}\, LI \qquad \frac{}{\Gamma | p \vdash p}\, AX$$

1. Eagerly invert *RI* to decompose implications in the RHS
2. When the RHS becomes atomic, apply *CTR* to decide a formula in the LHS to focus on
3. As long as the focus is an implication, apply *LI*, keeping the focus on the succedent formula, and launching sub-problems
4. When the focus becomes atomic, compare with the atom in the RHS; if equal, finish with *AX*, otherwise return to 2.

From *LJT* to LJT_∂

- Partial derivations: incomplete derivations found in the proof search process

From *LJT* to LJT_∂

- Partial derivations: incomplete derivations found in the proof search process
- Recall sequent forms of *LJT*:
  $\sigma ::= (\Gamma \vdash A) \quad \tau ::= (\Gamma | A \vdash p)$

From *LJT* to LJT_∂

- Partial derivations: incomplete derivations found in the proof search process
- Recall sequent forms of *LJT*:
  $\sigma ::= (\Gamma \vdash A) \quad \tau ::= (\Gamma | A \vdash p)$
- Formal sequents $\underline{\sigma} ::= (\underline{\Gamma \vdash A}) \qquad \underline{\tau} ::= (\underline{\Gamma | A \vdash p})$

From LJT to LJT_∂

- Partial derivations: incomplete derivations found in the proof search process
- Recall sequent forms of LJT:
  $\sigma ::= (\Gamma \vdash A) \quad \tau ::= (\Gamma | A \vdash p)$
- Formal sequents $\quad \underline{\sigma} ::= (\underline{\Gamma \vdash A}) \qquad \underline{\tau} ::= (\underline{\Gamma | A \vdash p})$
- Partial proof terms: proof terms with occurrences of formal sequents, they represent partial derivations

From LJT to LJT_∂

- Partial derivations: incomplete derivations found in the proof search process
- Recall sequent forms of LJT:
  $\sigma ::= (\Gamma \vdash A) \quad \tau ::= (\Gamma | A \vdash p)$
- Formal sequents $\quad \underline{\sigma} ::= (\underline{\Gamma \vdash A}) \qquad \underline{\tau} ::= (\underline{\Gamma | A \vdash p})$
- Partial proof terms: proof terms with occurrences of formal sequents, they represent partial derivations
- Reduction relation on partial proof terms expressing proof search, such that total terms are normal forms

From LJT to LJT_∂

- Partial derivations: incomplete derivations found in the proof search process
- Recall sequent forms of LJT:
  $\sigma ::= (\Gamma \vdash A) \quad \tau ::= (\Gamma | A \vdash p)$
- Formal sequents $\quad \underline{\sigma} ::= (\underline{\Gamma \vdash A}) \qquad \underline{\tau} ::= (\underline{\Gamma | A \vdash p})$
- Partial proof terms: proof terms with occurrences of formal sequents, they represent partial derivations
- Reduction relation on partial proof terms expressing proof search, such that total terms are normal forms
- Aim: proof search as normalization

$$\underbrace{LJT \text{ derives } \Gamma \vdash t : A}_{\text{proof search succeds}} \text{ iff } \underbrace{(\underline{\Gamma \vdash A}) \twoheadrightarrow t}_{\text{the formal sequent normalizes}}$$

LJT_∂: partial proof terms

(Partial proof terms)	t, u	$::=$	$\lambda x^A.t \mid x\hat{} l \mid \underline{\sigma}$
(Partial proof lists)	l	$::=$	$\star \mid u :: l \mid \underline{\tau}$
(Formal term sequents)	$\underline{\sigma}$	$::=$	$\underline{\Gamma \vdash A}$
(Formal list sequents)	$\underline{\tau}$	$::=$	$\underline{\Gamma \vert A \vdash p}$

LJT_∂: partial sequents

Partial sequents $\Xi \Vdash t : \sigma$ $\Xi \Vdash l : \tau$ (proof state)

- Ξ list of formal sequents (proof obligations)
- t (resp. l) partial proof term (resp. list) (record of the search)
- σ (resp. τ) (goal sequent)

LJT_∂: partial derivations

Inference rules

$$\frac{}{[\underline{\sigma}] \Vdash \underline{\sigma} : \sigma}\ \partial \qquad \frac{}{[\underline{\tau}] \Vdash \underline{\tau} : \tau}\ \partial$$

$$\frac{}{\epsilon \Vdash \star : (\Gamma | p \vdash p)}\ AX \qquad \frac{\Xi \Vdash l : (\Gamma, x : A | A \vdash p)}{\Xi \Vdash x\hat{}l : (\Gamma, x : A \vdash p)}\ CTR$$

$$\frac{\Xi \Vdash t : (\Gamma, x : A \vdash B)}{\Xi \Vdash \lambda x^A.t : (\Gamma \vdash A \supset B)}\ RI$$

$$\frac{\Xi_1 \Vdash u : (\Gamma \vdash A) \quad \Xi_2 \Vdash l : (\Gamma | B \vdash p)}{\Xi_1 @ \Xi_2 \Vdash u :: l : (\Gamma | A \supset B \vdash p)}\ LI$$

LJT_∂: reduction rules

$$
\begin{array}{rrcl}
(IR) & \underline{\Gamma \vdash A \supset B} & \rightarrow & \lambda x^A.(\underline{\Gamma, x : A \vdash B}) \\
(SFL) & \underline{\Gamma, x : A \vdash p} & \rightarrow & x\hat{}(\underline{\Gamma, x : A|A \vdash p}) \\
(KFL) & \underline{\Gamma|A \supset B \vdash C} & \rightarrow & (\underline{\Gamma \vdash A}) :: (\underline{\Gamma|B \vdash C}) \\
(FFL) & \underline{\Gamma|p \vdash p} & \rightarrow & \star
\end{array}
$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$\frac{\vdash (p \supset p) \supset p \supset p}{}$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$\rightarrow^2_{IR} \quad \frac{\vdash (p \supset p) \supset p \supset p}{\lambda f^{p \supset p} \lambda x^{p}.(\underline{\Gamma \vdash p})}$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$\begin{array}{ll} & \underline{\vdash (p \supset p) \supset p \supset p} \\ \rightarrow^2_{IR} & \lambda f^{p \supset p} \lambda x^p.(\underline{\Gamma \vdash p}) \\ \rightarrow_{SFL} & \lambda f^{p \supset p} \lambda x^p.f\hat{}(\underline{\Gamma | p \supset p \vdash p}) \end{array}$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$
\begin{array}{ll}
 & \underline{\vdash (p \supset p) \supset p \supset p} \\
\rightarrow^2_{IR} & \lambda f^{p \supset p} \lambda x^p.(\underline{\Gamma \vdash p}) \\
\rightarrow_{SFL} & \lambda f^{p \supset p} \lambda x^p.f\hat{\ }(\underline{\Gamma | p \supset p \vdash p}) \\
\rightarrow_{KFL} & \lambda f^{p \supset p} \lambda x^p.f\hat{\ }\Big((\underline{\Gamma \vdash p}) :: (\underline{\Gamma | p \vdash p})\Big)
\end{array}
$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$
\begin{aligned}
& \underline{\vdash (p \supset p) \supset p \supset p} \\
\rightarrow^{2}_{IR}\quad & \lambda f^{p \supset p} \lambda x^{p}.(\underline{\Gamma \vdash p}) \\
\rightarrow_{SFL}\quad & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}(\underline{\Gamma | p \supset p \vdash p}) \\
\rightarrow_{KFL}\quad & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}\left((\underline{\Gamma \vdash p}) :: (\underline{\Gamma | p \vdash p})\right) \\
\rightarrow_{FFL}\quad & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}\left((\underline{\Gamma \vdash p}) :: \star\right)
\end{aligned}
$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$
\begin{array}{ll}
 & \underline{\vdash (p \supset p) \supset p \supset p} \\
\to^2_{IR} & \lambda f^{p \supset p} \lambda x^p . (\underline{\Gamma \vdash p}) \\
\to_{SFL} & \lambda f^{p \supset p} \lambda x^p . f\hat{}(\underline{\Gamma | p \supset p \vdash p}) \\
\to_{KFL} & \lambda f^{p \supset p} \lambda x^p . f\hat{}\left((\underline{\Gamma \vdash p}) :: (\underline{\Gamma | p \vdash p})\right) \\
\to_{FFL} & \lambda f^{p \supset p} \lambda x^p . f\hat{}\left((\underline{\Gamma \vdash p}) :: \star\right) \\
\to_{SFL} & \lambda f^{p \supset p} \lambda x^p . f\hat{}\left((x\hat{}(\underline{\Gamma | p \vdash p})) :: \star\right)
\end{array}
$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$
\begin{array}{ll}
 & \underline{\vdash (p \supset p) \supset p \supset p} \\
\to^2_{IR} & \lambda f^{p \supset p} \lambda x^p . (\underline{\Gamma \vdash p}) \\
\to_{SFL} & \lambda f^{p \supset p} \lambda x^p . f \hat{} (\underline{\Gamma | p \supset p \vdash p}) \\
\to_{KFL} & \lambda f^{p \supset p} \lambda x^p . f \hat{} \left((\underline{\Gamma \vdash p}) :: (\underline{\Gamma | p \vdash p}) \right) \\
\to_{FFL} & \lambda f^{p \supset p} \lambda x^p . f \hat{} \left((\underline{\Gamma \vdash p}) :: \star \right) \\
\to_{SFL} & \lambda f^{p \supset p} \lambda x^p . f \hat{} \left((x \hat{} (\underline{\Gamma | p \vdash p})) :: \star \right) \\
\to_{FFL} & \lambda f^{p \supset p} \lambda x^p . f \hat{} ((x \hat{} \star) :: \star)
\end{array}
$$

Example

$\vdash (p \supset p) \supset p \supset p$

Below let $\Gamma := \{f : p \supset p, x : p\}$

$$
\begin{array}{ll}
 & \dfrac{\vdash (p \supset p) \supset p \supset p}{\lambda f^{p \supset p} \lambda x^{p}.(\underline{\Gamma \vdash p})} \\
\rightarrow^{2}_{IR} & \\
\rightarrow_{SFL} & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}(\underline{\Gamma | p \supset p \vdash p}) \\
\rightarrow_{KFL} & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}\left((\underline{\Gamma \vdash p}) :: (\underline{\Gamma | p \vdash p})\right) \\
\rightarrow_{FFL} & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}\left((\underline{\Gamma \vdash p}) :: \star\right) \\
\rightarrow_{SFL} & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}\left((x\hat{}(\underline{\Gamma | p \vdash p})) :: \star\right) \\
\rightarrow_{FFL} & \lambda f^{p \supset p} \lambda x^{p}.f\hat{}((x\hat{}\star) :: \star)
\end{array}
$$

Cf $\lambda f^{p \supset p} \lambda x^{p}.fx$ $(\equiv \overline{1})$

Results

Proposition (Conservativity)

1. LJT derives $\Gamma \vdash t : A$ iff LJT_∂ derives $\epsilon \Vdash t : (\Gamma \vdash A)$.
2. LJT derives $\Gamma | A \vdash l : p$ iff LJT_∂ derives $\epsilon \Vdash l : (\Gamma | A \vdash p)$.

Lemma (Record of search)

1. If LJT_∂ derives $\Xi \Vdash t : \sigma$ then $\underline{\sigma} \twoheadrightarrow t$.
2. If LJT_∂ derives $\Xi \Vdash l : \tau$ then $\underline{\tau} \twoheadrightarrow t$.

Proposition (Subject reduction)

1. If $\Xi \Vdash t : \sigma$ and $t \to t'$, then $\Xi' \Vdash t' : \sigma$, for some Ξ'.
2. If $\Xi \Vdash l : \tau$ and $l \to l'$, then $\Xi' \Vdash l' : \tau$, for some Ξ'.

Main result

Theorem (Proof search as normalization)

1. *LJT* derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. *LJT* derives $\Gamma | A \vdash l : B$ iff $\underline{\Gamma | A \vdash p} \twoheadrightarrow l$.

Main result

Theorem (Proof search as normalization)

1. *LJT* derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. *LJT* derives $\Gamma|A \vdash l : B$ iff $\underline{\Gamma|A \vdash p} \twoheadrightarrow l$.

Proof.

LJT derives $\Gamma \vdash t : A$

Main result

Theorem (Proof search as normalization)

1. LJT derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. LJT derives $\Gamma|A \vdash l : B$ iff $\underline{\Gamma|A \vdash p} \twoheadrightarrow l$.

Proof.

$$\begin{array}{ll} & LJT \text{ derives } \Gamma \vdash t : A \\ \Rightarrow & LJT_{\partial} \text{ derives } \epsilon \Vdash t : (\Gamma \vdash A) \quad \text{(by conservativity)} \end{array}$$

Main result

Theorem (Proof search as normalization)

1. LJT derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. LJT derives $\Gamma | A \vdash l : B$ iff $\underline{\Gamma | A \vdash p} \twoheadrightarrow l$.

Proof.

$$\begin{array}{lll} & LJT \text{ derives } \Gamma \vdash t : A & \\ \Rightarrow & LJT_\partial \text{ derives } \epsilon \Vdash t : (\Gamma \vdash A) & \text{(by conservativity)} \\ \Rightarrow & \underline{\Gamma \vdash A} \twoheadrightarrow t & \text{(by record of search)} \end{array}$$

Main result

Theorem (Proof search as normalization)

1. LJT derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. LJT derives $\Gamma | A \vdash l : B$ iff $\underline{\Gamma | A \vdash p} \twoheadrightarrow l$.

Proof.

$$\begin{array}{lll} & LJT \text{ derives } \Gamma \vdash t : A & \\ \Rightarrow & LJT_{\partial} \text{ derives } \epsilon \Vdash t : (\Gamma \vdash A) & \text{(by conservativity)} \\ \Rightarrow & \underline{\Gamma \vdash A} \twoheadrightarrow t & \text{(by record of search)} \end{array}$$

$$[\underline{\Gamma \vdash A}] \Vdash \underline{\Gamma \vdash A} : (\Gamma \vdash A) \quad \text{(by rule } \partial)$$

Main result

Theorem (Proof search as normalization)

1. *LJT* derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. *LJT* derives $\Gamma | A \vdash l : B$ iff $\underline{\Gamma | A \vdash p} \twoheadrightarrow l$.

Proof.

$$
\begin{array}{lll}
 & LJT \text{ derives } \Gamma \vdash t : A & \\
\Rightarrow & LJT_\partial \text{ derives } \epsilon \Vdash t : (\Gamma \vdash A) & \text{(by conservativity)} \\
\Rightarrow & \underline{\Gamma \vdash A} \twoheadrightarrow t & \text{(by record of search)}
\end{array}
$$

$$
\begin{array}{lll}
 & [\underline{\Gamma \vdash A}] \Vdash \underline{\Gamma \vdash A} : (\Gamma \vdash A) & \text{(by rule } \partial) \\
\Rightarrow & \Xi \Vdash t : (\Gamma \vdash A) & \text{(by subject reduction)}
\end{array}
$$

Main result

Theorem (Proof search as normalization)

1. LJT derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. LJT derives $\Gamma | A \vdash l : B$ iff $\underline{\Gamma | A \vdash p} \twoheadrightarrow l$.

Proof.

$$
\begin{array}{lll}
 & LJT \text{ derives } \Gamma \vdash t : A & \\
\Rightarrow & LJT_\partial \text{ derives } \epsilon \Vdash t : (\Gamma \vdash A) & \text{(by conservativity)} \\
\Rightarrow & \underline{\Gamma \vdash A} \twoheadrightarrow t & \text{(by record of search)}
\end{array}
$$

$$
\begin{array}{lll}
 & [\underline{\Gamma \vdash A}] \Vdash \underline{\Gamma \vdash A} : (\Gamma \vdash A) & \text{(by rule } \partial) \\
\Rightarrow & \Xi \Vdash t : (\Gamma \vdash A) & \text{(by subject reduction)} \\
\Rightarrow & \epsilon \Vdash t : (\Gamma \vdash A) & \text{(since } t \text{ is total)}
\end{array}
$$

Main result

Theorem (Proof search as normalization)

1. *LJT* derives $\Gamma \vdash t : A$ iff $\underline{\Gamma \vdash A} \twoheadrightarrow t$.
2. *LJT* derives $\Gamma | A \vdash l : B$ iff $\underline{\Gamma | A \vdash p} \twoheadrightarrow l$.

Proof.

$$\begin{array}{lll} & LJT \text{ derives } \Gamma \vdash t : A & \\ \Rightarrow & LJT_\partial \text{ derives } \epsilon \Vdash t : (\Gamma \vdash A) & \text{(by conservativity)} \\ \Rightarrow & \underline{\Gamma \vdash A} \twoheadrightarrow t & \text{(by record of search)} \end{array}$$

$$\begin{array}{lll} & [\underline{\Gamma \vdash A}] \Vdash \underline{\Gamma \vdash A} : (\Gamma \vdash A) & \text{(by rule } \partial) \\ \Rightarrow & \Xi \Vdash t : (\Gamma \vdash A) & \text{(by subject reduction)} \\ \Rightarrow & \epsilon \Vdash t : (\Gamma \vdash A) & \text{(since } t \text{ is total)} \\ \Rightarrow & LJT \text{ derives } \Gamma \vdash t : A & \text{(by conservativity)} \end{array}$$

□

NJT

NJT

Sequents $\sigma ::= \Gamma \vdash A \quad \rho ::= \Gamma \rhd A$

Inference rules

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B}\ I \qquad \frac{}{\Gamma, x : A \rhd A}\ A$$

$$\frac{\Gamma \rhd A \supset B \quad \Gamma \vdash A}{\Gamma \rhd B}\ E \qquad \frac{\Gamma \rhd p}{\Gamma \vdash p}\ C$$

NJT with proof terms

Proof terms and head terms

$$\begin{array}{rcl} M, N & ::= & \lambda x^A.M \mid app(H) \\ H & ::= & x \mid HN \end{array}$$

Sequents $\Gamma \vdash M : A$ and $\Gamma \rhd H : A$

Inference rules

$$\frac{}{\Gamma, x : A \rhd x : A}\ A \qquad \frac{\Gamma \rhd H : p}{\Gamma \vdash app(H) : p}\ C$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \supset B}\ I \qquad \frac{\Gamma \rhd H : A \supset B \quad \Gamma \vdash N : A}{\Gamma \rhd HN : B}\ E$$

NJT: proof search (intercalation)

Inference rules

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash A \supset B}\ I \qquad \frac{}{\Gamma, x : A \rhd A}\ A$$

$$\frac{\Gamma \rhd A \supset B \quad \Gamma \vdash A}{\Gamma \rhd B}\ E \qquad \frac{\Gamma \rhd p}{\Gamma \vdash p}\ C$$

1. Implications in the RHS are decomposed by the bottom-up application of rule I
2. When an atom p is computed, choose an assumption (rule A) and start the top-down phase
3. Decompose implications by top-down application of rule E, launching sub-problems
4. If the atom computed top-down is p, then finish with rule C, else return to 2.

NJT_∂: partial proof terms

(Partial proof terms)	M, N	$::=$	$\lambda x^A.M \mid app(H)$
		$\mid$	$\underline{\sigma} \mid \mathsf{app}(H, \underline{\rho}, p)$
(Partial head terms)	H	$::=$	$x \mid HN$
(Formal term sequents)	$\underline{\sigma}$	$::=$	$\underline{\Gamma \vdash A}$
(Formal head sequents)	$\underline{\rho}$	$::=$	$\underline{\Gamma \rhd H : A}$

NJT_∂: partial derivations

Sequents $\quad \Xi \Vdash M : \sigma \qquad \Xi \Vdash H : \rho$

Inference rules

$$\frac{}{[\underline{\sigma}] \Vdash \underline{\sigma} : \sigma}\ \partial \qquad \frac{\Xi \Vdash H : \rho \quad \rho = (\Gamma \rhd A)}{\Xi @ [\underline{\rho}] \Vdash \mathsf{app}(H, \underline{\rho}, p) : (\Gamma \vdash p)}\ \partial$$

$$\frac{}{\epsilon \Vdash x : (\Gamma, x : A \rhd A)}\ A \qquad \frac{\Xi \Vdash H : (\Gamma \rhd p)}{\Xi \Vdash app(H) : (\Gamma \vdash p)}\ C$$

$$\frac{\Xi \Vdash M : (\Gamma, x : A \vdash B)}{\Xi \Vdash \lambda x^A.M : (\Gamma \vdash A \supset B)}\ I$$

$$\frac{\Xi_1 \Vdash H : (\Gamma \rhd A \supset B) \quad \Xi_2 \Vdash N : (\Gamma \vdash A)}{\Xi_1 @ \Xi_2 \Vdash HN : (\Gamma \rhd B)}\ E$$

NJT_∂: reduction rules

$$
\begin{array}{rrcl}
(IR) & \underline{\Gamma \vdash A \supset B} & \rightarrow & \lambda x^A.(\underline{\Gamma, x : A \vdash B}) \\
(SFD) & \underline{\Gamma, x : A \vdash p} & \rightarrow & \mathrm{app}(x, \underline{\Gamma, x : A \rhd A}, p) \\
(KFD) & \mathrm{app}(H, \underline{\Gamma \rhd A \supset B}, p) & \rightarrow & \mathrm{app}(H(\underline{\Gamma \vdash A}), \underline{\Gamma \rhd B}, p) \\
(FFD) & \mathrm{app}(H, \underline{\Gamma \rhd p}, p) & \rightarrow & app(H)
\end{array}
$$

Focusing vs intercalation

$LJT \cong NJT$

- Proof terms of LJT and NJT are in bijective correspondence
- $\hat{x}(u_1 :: (u_2 :: \star))$ corresponds to $app((xN_1)N_2)$
- Let $\Theta : LJT \to NJT$ be this bijection with inverse Ψ
- These maps are sound, they lift to a sound bijection of proofs

$LJT \cong NJT$

- Proof terms of LJT and NJT are in bijective correspondence
- $\hat{x}(u_1 :: (u_2 :: \star))$ corresponds to $app((xN_1)N_2)$
- Let $\Theta : LJT \to NJT$ be this bijection with inverse Ψ
- These maps are sound, they lift to a sound bijection of proofs

$LJT_\partial \cong NJT_\partial$

- The bijections Θ and Ψ extend to partial proof terms
- By soundness they lift to sound bijection of partial proofs
- New: they establish an isomorphism between the rewriting relations

Translations $\Theta : LJT_\partial \to NJT_\partial$

$\Theta(t) = M$

$$\begin{array}{rcl} \Theta(\lambda x.t) & = & \lambda x.\Theta t \\ \Theta(x \hat{} l) & = & \Theta(x, l) \\ \Theta(\underline{\sigma}) & = & \underline{\sigma} \end{array}$$

$\Theta(H, l) = M$

$$\begin{array}{rcl} \Theta(H, u :: l) & = & \Theta(H(\Theta u), l) \\ \Theta(H, \star) & = & app(H) \\ \Theta(H, \underline{\Gamma|A \vdash p}) & = & \mathrm{app}(H, \underline{\Gamma \rhd A}, p) \end{array}$$

Translations $\Psi : NJT_\partial \to LJT_\partial$

$\Psi(M) = t$

$$\begin{array}{rcl}
\Psi(\lambda x.M) & = & \lambda x.\Psi M \\
\Psi(app(H)) & = & \Psi(H, \star) \\
\Psi(\underline{\sigma}) & = & \underline{\sigma} \\
\Psi(\mathrm{app}(H, \underline{\Gamma \triangleright A}, p)) & = & \Psi(H, \underline{\Gamma | A \vdash p})
\end{array}$$

$\Psi(H, l) = t$

$$\begin{array}{rcl}
\Psi(HN, l) & = & \Psi(H, (\Psi N) :: l) \\
\Psi(x, l) & = & x\hat{\ }l
\end{array}$$

Soundness

Soundness of Θ

$$\frac{\Xi \Vdash t : \sigma}{\Xi' \Vdash \Theta t : \sigma} \qquad \frac{\Xi_1 \Vdash H : (\Gamma \rhd A) \quad \Xi_2 \Vdash l : (\Gamma | A \vdash p)}{\Xi_1 @ \Xi_2' \Vdash \Theta(H, l) : (\Gamma \vdash p)}$$

Soundness of Ψ

$$\frac{\Xi \Vdash M : \sigma}{\Xi' \Vdash \Psi M : \sigma} \qquad \frac{\Xi_1 \Vdash H : (\Gamma \rhd A) \quad \Xi_2 \Vdash l : (\Gamma | A \vdash p)}{\Xi_1' @ \Xi_2 \Vdash \Psi(H, l) : (\Gamma \vdash p)}$$

$LJT_\partial \cong NJT_\partial$

Theorem

1. $\Theta\Psi M = M$ and $\Psi\Theta t = t$.
2. $t \to t'$ in LJT_∂ iff $\Theta t \to \Theta t'$ in NJT_∂.
3. $M \to M'$ in NJT_∂ iff $\Psi M \to \Psi M'$ in LJT_∂.

1. Contributions

1. Contributions
    - Modeling of proof search at a level of abstraction that still belongs to proof theory
    - *NJT* as a reformulation of Sieg's intercalation calculus
    - Folklore (?) theorem: focusing equivalent to intercalation

1. Contributions
   - Modeling of proof search at a level of abstraction that still belongs to proof theory
   - *NJT* as a reformulation of Sieg's intercalation calculus
   - Folklore (?) theorem: focusing equivalent to intercalation
2. Related work
   - Type theories with meta-variables and explicit substitutions (*e. g.* Muñoz 2001, Nanevski *et al* 2008)
   - Open proofs and open terms (Geuvers-Jojgov 2002)
   - Specification in rewriting logic (Olarte *et al* 2023)

1. Contributions
   - Modeling of proof search at a level of abstraction that still belongs to proof theory
   - *NJT* as a reformulation of Sieg's intercalation calculus
   - Folklore (?) theorem: focusing equivalent to intercalation
2. Related work
   - Type theories with meta-variables and explicit substitutions (*e. g.* Muñoz 2001, Nanevski *et al* 2008)
   - Open proofs and open terms (Geuvers-Jojgov 2002)
   - Specification in rewriting logic (Olarte *et al* 2023)
3. Ongoing work
   - Extend the results beyond the toy case studies
   - Approach less idealized proof search

THANK YOU