Reusing Learning Objects via Theory Morphisms

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Learning Objects

What is a Learning Object?

Theorem. Propositional logic $\langle \mathcal{L}_{\mathrm{PL}^0}, \mathcal{K}_{\mathrm{PL}^0}, \models_{\mathrm{PL}^0} \rangle$ naturally forms a logical system $\langle \mathcal{L}, \mathcal{K}, \models \rangle$.

Definition. Let A be a formula of propositional logic. A is satisfiable iff there exists a model $\mathcal{I}$ such that $\mathcal{I}(A) = \mathsf{T}$.

Exercise. Is the formula $p \wedge (q \Rightarrow \neg p)$ satisfiable?
☐ Yes
☐ No

Generalization and Exemplification

Definition. A logical system is a triple $\langle \mathcal{L}, \mathcal{K}, \models \rangle$, where

- $\mathcal{L}$ is a formal language, whose elements are called formulas,
- $\mathcal{K}$ is a set, whose elements are called models,
- $\models \subseteq \mathcal{K} \times \mathcal{L}$.

Definition. Propositional logic is the triple $\langle \mathcal{L}_{\mathrm{PL}^0}, \mathcal{K}_{\mathrm{PL}^0}, \models_{\mathrm{PL}^0} \rangle$, where

- $\mathcal{L}_{\mathrm{PL}^0}$ is the set of all propositional formulas,
- $\mathcal{K}_{\mathrm{PL}^0}$ is the set of all Boolean interpretation functions,
- $\mathcal{I} \models_{\mathrm{PL}^0} A$ iff $\mathcal{I}(A) = \mathsf{T}$.

Theorem. Propositional logic $\langle \mathcal{L}_{\mathrm{PL}^0}, \mathcal{K}_{\mathrm{PL}^0}, \models_{\mathrm{PL}^0} \rangle$ naturally forms a logical system $\langle \mathcal{L}, \mathcal{K}, \models \rangle$.

Recontextualizing Statements

Logical Systems:

Definition. Let F be a formula of a logical system. A model $\mathcal{M}$ satisfies F iff $\mathcal{M} \models F$.

$$\Downarrow$$

Propositional Logic:

Definition. Let A be a formula of propositional logic. A model $\mathcal{I}$ satisfies A iff $\mathcal{I} \models_{\mathrm{PL}^0} A$.

Definition. Let A be a formula of propositional logic. A model $\mathcal{I}$ satisfies A iff $\mathcal{I}(A) = \mathsf{T}$.

Recontextualizing Exercises

Logical Systems:

Exercise. Is the formula $F :=$ `definiens(`F`)` satisfiable?
☐ Yes
☐ No
Feedback: Actually, there is a model $\mathcal{M}$ that satisfies F:
`definiens(`$\mathcal{M}$`)`. Then `conclusion(`Φ`)`:
Indeed, `proof(`Φ`)`.

Propositional Logic:

Exercise. Is the formula $A := p \wedge (q \Rightarrow \neg p)$ satisfiable?
☐ Yes
☐ No
Feedback: Actually, there is a model $\mathcal{I}$ that satisfies A:
It maps p to T and q to F. Then $\mathcal{I} \models_{\mathrm{PL}^0} A$:
Indeed, this can directly be seen by evaluating the truth table for A.

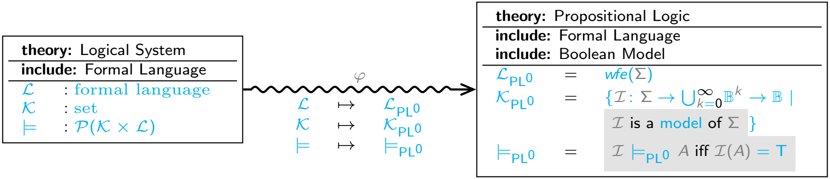
Theory Morphisms

What is a Theory Morphism?

Definition. A theory morphism $\varphi : S \rightsquigarrow T$ between two theories S and T is a mapping of the symbols in S to expressions in T such that

$$s : \tau \quad \Longrightarrow \quad \varphi(s) : \varphi(\tau)$$

for all symbols s and types τ in S.



Representing Theory Morphisms in sTeX

```
\begin{mathstructure}{logsys}[logical system]
  ...
  % The satisfaction relation:
  \symdef{satrel}[
    args=2,
    type={\powerset{\cart{\modcls,\flang}}}
  ]{#1\vDash#2}
  ...
\end{mathstructure}
```

```
\begin{mathstructure}{proplog}[propositional logic]
  ...
  % The satisfaction relation:
  \symdef{psat}[
    args=2,
    def={\textrm{$\psat{\Ivar!}{\Avar}$ iff $\eq{\Ivar{\Avar},\semtrue}$}}
  ]{#1\vDash_{\textrm{PL}^0}#2}
  ...

  % The theory morphism from logical systems to propositional logic:
  \begin{realization}{logsys}
    ...
    \assign{satrel}{\psat!}
    ...
  \end{realization}
\end{mathstructure}
```

Presenting Theory Morphisms

theory: Logical System
include: Formal Language
$\mathcal{L}$: formal language
$\mathcal{K}$: set
$\models$: $\mathcal{P}(\mathcal{K} \times \mathcal{L})$

φ:
$\mathcal{L} \mapsto \mathcal{L}_{\mathrm{PL}^0}$
$\mathcal{K} \mapsto \mathcal{K}_{\mathrm{PL}^0}$
$\models \mapsto \models_{\mathrm{PL}^0}$

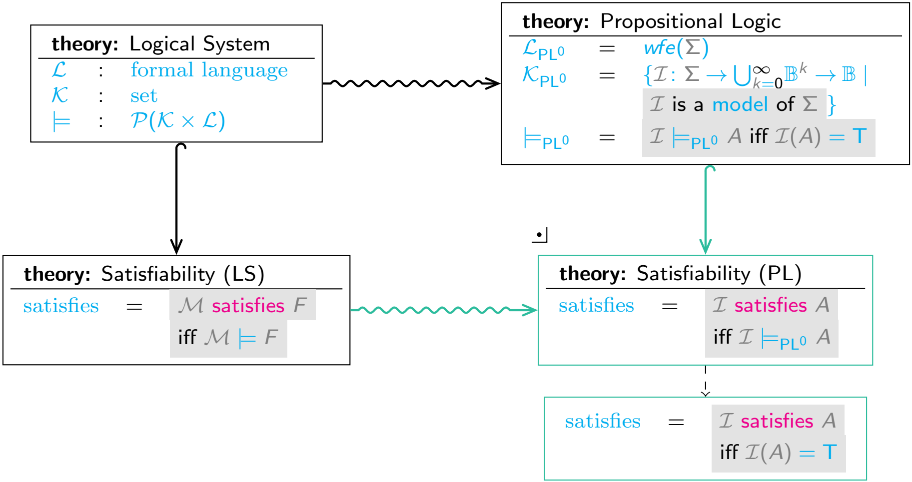
theory: Propositional Logic
include: Formal Language
include: Boolean Model
$\mathcal{L}_{\mathrm{PL}^0} = wfe(\Sigma)$
$\mathcal{K}_{\mathrm{PL}^0} = \{\mathcal{I} \colon \Sigma \to \bigcup_{k=0}^{\infty} \mathbb{B}^k \to \mathbb{B} \mid$ $\mathcal{I}$ is a model of Σ $\}$
$\models_{\mathrm{PL}^0}$ = $\mathcal{I} \models_{\mathrm{PL}^0} A$ iff $\mathcal{I}(A) = \mathrm{T}$

Theorem. ⟨target concept⟩ ⟨target structure⟩ naturally forms a ⟨source concept⟩ ⟨source structure⟩.

Theorem. Propositional logic $\langle \mathcal{L}_{\mathrm{PL}^0}, \mathcal{K}_{\mathrm{PL}^0}, \models_{\mathrm{PL}^0} \rangle$ naturally forms a logical system $\langle \mathcal{L}, \mathcal{K}, \models \rangle$.

Recontextualizing Learning Objects Along Theory Morphisms

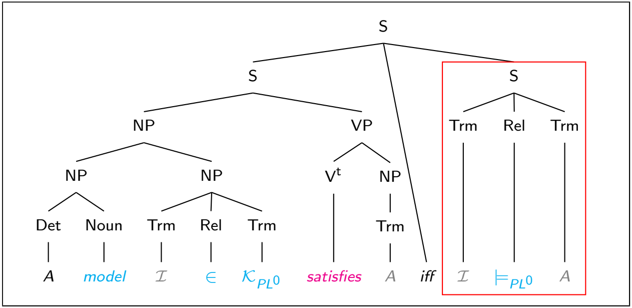
Recontextualizing Statements (Revisited)

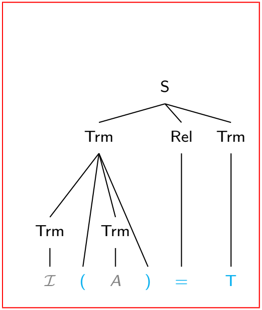


Definition Expansion

"A model $\mathcal{I} \in \mathcal{K}_{\mathrm{PL}^0}$ satisfies A iff $\mathcal{I} \models_{\mathrm{PL}^0} A$."

$$\Downarrow$$





$$\Downarrow$$

"A model $\mathcal{I} \in \mathcal{K}_{\mathrm{PL}^0}$ satisfies A iff $\mathcal{I}(A) = \mathrm{T}$."

Recontextualizing Exercises (Revisited)

theory: Exercise (LS)

F	:	formula	
$\mathcal{M}$	:	model	
Φ	:	$\vdash \mathcal{M} \models F$	

ψ

F	$\mapsto$	A
$\mathcal{M}$	$\mapsto$	$\mathcal{I}$
Φ	$\mapsto$	Φ_{PL^0}

theory: Exercise (PL)

p	:	constant		
q	:	constant		
A	:	formula	=	$p \wedge (q \Rightarrow \neg p)$
$\mathcal{I}$	:	model	=	it maps p to T and q to F
Φ_{PL^0}	:	$\vdash \mathcal{I} \models_{PL^0} A$	=	this can directly be seen by evaluating the truth table for A

Exercise. Is the formula $F :=$ definiens(F) satisfiable?
☐ Yes
☐ No
Feedback: Actually, there is a model $\mathcal{M}$ that satisfies F:
definiens($\mathcal{M}$). Then conclusion(Φ):
Indeed, proof(Φ).

Conclusion and Future Work

Reusing Learning Objects via Theory Morphisms

- We can **automatically** ...
  - ... present theory morphisms in natuaral language.
  - ... recontextualize statements along theory morphisms.
  - ... recontextualize exercises with solutions and feedback along theory morphisms.

⟶ https://gitos.rrze.fau.de/voll-ki/fau/system/relocalization/

- We have a growing corpus of theory morphisms and templates that are suited for automatic recontextualization.

⟶ E.g. https://gl.mathhub.info/courses/FAU/AI/problems/-/tree/main/source/csp/prob

- **Future Work:**
  - Properly handling the intricacies of natural language.
  - Integrating the recontextualization processes in our adaptive learning assistant ALeA.

  ⟶ https://courses.voll-ki.fau.de/