Towards Mac Lane's Comparison Theorem for the (co)Kleisli Construction in Coq

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Abstract

This short paper summarizes an ongoing work on the formalization of Mac Lane's comparison theorem for the (co)Kleisli construction in the Coq proof assistant.

1 Adjoint Functors and Monads

Definition 1.1. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A *hom-adjunction* $F \dashv G\colon \mathscr{D} \to \mathscr{C}$ is a triple $\langle F, G, \varphi\rangle$ such that $F\colon \mathscr{C} \to \mathscr{D}$, $G\colon \mathscr{D} \to \mathscr{C}$ are functors and $\varphi = (\varphi_{X,\,A})_{X,\,A}$ is a family of bijections, natural in X and A, where X is an object of $\mathscr{C}$ and A is an object of $\mathscr{D}$:

$$\varphi_{X,\,A}\colon Hom_{\mathscr{D}}(FX,\,A) \xrightarrow{\cong} Hom_{\mathscr{C}}(X,\,GA) \tag{1}$$

Let $F \dashv G\colon \mathscr{D} \to \mathscr{C}$ be a hom-adjunction. By instantiating $A = FX$ in (1) we obtain $\eta_X\colon X \to GFX$ in $\mathscr{C}$ which is the image of id_{FX} by $\varphi_{X,\,FX}$. Symmetrically, by setting $X = GA$, we obtain $\varepsilon_A\colon FGA \to A$ in $\mathscr{D}$ which is the image of id_{GA} by $\varphi^{-1}_{GA,\,A}$. As shown in [ML71, Ch. IV, §1], $\eta\colon Id_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow Id_{\mathscr{D}}$ are *natural transformations*. This gives us the following proposition by [ML71, Ch. VI, §1] and [ML71, Ch. IV, §1, Theorem 1].

Proposition 1.2. *A hom-adjunction $F \dashv G\colon \mathscr{D} \to \mathscr{C}$, with associated family of bijections φ as in* Definition 1.1*, determines a monad on $\mathscr{C}$ and a comonad on $\mathscr{D}$ as follows:*

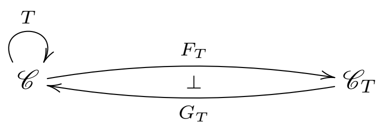
- *The monad (T, η, μ) on $\mathscr{C}$ has endofunctor $T = GF\colon \mathscr{C} \to \mathscr{C}$, unit $\eta\colon Id_{\mathscr{C}} \Rightarrow T$ where $\eta_X = \varphi_{X,\,FX}(id_{FX})$ and multiplication $\mu\colon T^2 \Rightarrow T$ such that $\mu_X = G(\varepsilon_{FX})$.*
- *The comonad (D, ε, δ) on $\mathscr{D}$ has endofunctor $D = FG\colon \mathscr{D} \to \mathscr{D}$, counit $\varepsilon\colon D \Rightarrow Id_{\mathscr{D}}$ where $\varepsilon_A = \varphi^{-1}_{GA,\,A}(id_{GA})$ and comultiplication $\delta\colon D \Rightarrow D^2$ such that $\delta_A = F(\eta_{GA})$.*

In addition, we have:

$$\varphi_{X,\,A} f = Gf \circ \eta_X\colon X \to GA \text{ for each } f\colon FX \to A \tag{2}$$

$$\varphi^{-1}_{X,\,A} g = \varepsilon_A \circ Fg\colon FX \to A \text{ for each } g\colon X \to GA. \tag{3}$$

Proposition 1.3. *Each monad (T, η, μ) on a category $\mathscr{C}$ determines a Kleisli category $\mathscr{C}_T$ and an associated hom-adjunction $F_T \dashv G_T\colon \mathscr{C}_T \to \mathscr{C}$ as follows:*



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We give an overview below, please find the related details in [ML71, Ch. VI, §5].

- *The categories $\mathscr{C}$ and $\mathscr{C}_T$ have the same objects and there is a morphism $f^\flat\colon X \to Y$ in $\mathscr{C}_T$ for each morphism $f\colon X \to TY$ in $\mathscr{C}$. So that there is a bijection defined as:*

$$(\varphi_T)_{X,Y}\colon Hom_{\mathscr{C}_T}(X,Y) \xrightarrow{\cong} Hom_{\mathscr{C}}(X,TY)$$
$$f^\flat \leftarrow\!\shortmid f$$

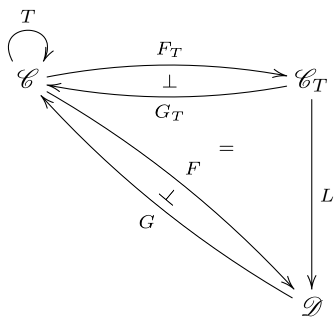
- *For each object X in $\mathscr{C}_T$, the identity arrow is $id_X = h^\flat\colon X \to X$ in $\mathscr{C}_T$ where $h = \eta_X\colon X \to TX$ in $\mathscr{C}$.*
- *The composition of a pair of morphisms $f^\flat\colon X \to Y$ and $g^\flat\colon Y \to Z$ in $\mathscr{C}_T$ is given by the Kleisli composition: $g^\flat \circ f^\flat = h^\flat\colon X \to Z$ where $h = \mu_Z \circ Tg \circ f\colon X \to TZ$ in $\mathscr{C}$.*
- *The functor $F_T\colon \mathscr{C} \to \mathscr{C}_T$ is the identity on objects. On morphisms,*

$$F_T f = (\eta_Y \circ f)^\flat, \text{ for each } f\colon X \to Y \text{ in } \mathscr{C}. \tag{4}$$

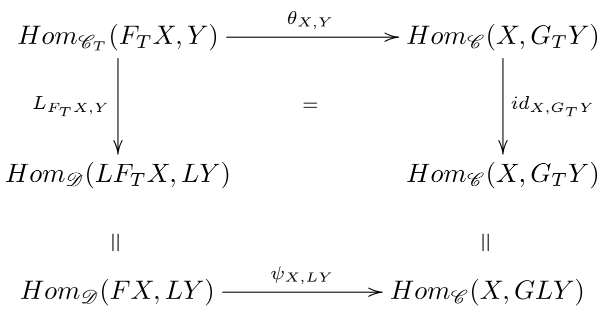
- *The functor $G_T\colon \mathscr{C}_T \to \mathscr{C}$ maps each object X in $\mathscr{C}_T$ to TX in $\mathscr{C}$. On morphisms,*

$$G_T(g^\flat) = \mu_Y \circ Tg, \text{ for each } g^\flat\colon X \to Y \text{ in } \mathscr{C}_T. \tag{5}$$

Theorem 1.4. (The comparison theorem for the Kleisli construction) *Let $F \dashv G\colon \mathscr{D} \to \mathscr{C}$ be a hom-adjunction and let (T,η,μ) be the associated monad on $\mathscr{C}$. Then, there is a unique* comparison functor *$L\colon \mathscr{C}_T \to \mathscr{D}$ such that $GL = G_T$ and $LF_T = F$, where $\mathscr{C}_T$ is the Kleisli category of (T,η,μ), with the associated hom-adjunction $F_T \dashv G_T\colon \mathscr{C}_T \to \mathscr{C}$.*



Proof. Let us first assume that $L\colon \mathscr{C}_T \to \mathscr{D}$ is a functor satisfying $GL = G_T$ and $LF_T = F$. And, let $\theta_{X,Y}\colon Hom_{\mathscr{C}_T}(F_T X, Y) \xrightarrow{\cong} Hom_{\mathscr{C}}(X, G_T Y)$ be a bijection associated to the hom-adjunction $F_T \dashv G_T$. Similarly, let $\psi_{X,Y}\colon Hom_{\mathscr{D}}(FX,Y) \xrightarrow{\cong} Hom_{\mathscr{C}}(X,GY)$ be a bijection associated to the hom-adjunction $F \dashv G$. Since both units of $F_T \dashv G_T$ and $F \dashv G$ are the unit η of the monad (T,η,μ) by [ML71, Ch. IV, §7, Proposition 1], we obtain the commutative diagram below:



Therefore, $L_{F_T X,Y} = \psi^{-1}_{X,LY} \circ \theta_{X,Y}$. This formula ensures that the functor L is *unique*. Using the Equation (2) in Proposition 1.2, we have: $\theta_{X,Y} f^\flat = G_T f^\flat \circ \eta_X\colon X \to G_T Y$, for each $f^\flat\colon F_T X = X \to Y$ in $\mathscr{C}_T$. Since $G_T f^\flat = \mu_Y \circ Tf$ in $\mathscr{C}$, for each $f^\flat\colon X \to Y$ in $\mathscr{C}_T$, by Equation (5), we have $\theta_{X,Y} f^\flat = \mu_Y \circ Tf \circ \eta_X\colon X \to$

$G_T F_T Y = G_T Y$. Thanks to the naturality of η, we get $\theta_{X,Y} f^\flat = \mu_Y \circ \eta_{TY} \circ f$. The monadic axiom $\mu_Y \circ \eta_{TY} = id_{TY}$ yields $\theta_{X,Y} f^\flat = f\colon X \to G_T Y$. Presumed that $G_T = GL$ and since F_T is the identity on objects, we have $\theta_{X,Y} f^\flat = f\colon X \to GLY$ and $LF_T Y = LY = FY$. Now, by Equation (3) in Proposition 1.2, we obtain $\psi^{-1}_{X,LY} f = \varepsilon_{LY} \circ Ff = \varepsilon_{FY} \circ Ff = \psi^{-1}_{X,FY} f$ for each $f\colon X \to GFY$ in $\mathscr{C}$. Hence $\psi^{-1}_{X,LY}(\theta_{X,Y} f^\flat) = \psi^{-1}_{X,FY} f = \varepsilon_{FY} \circ Ff$.

In other words: given a functor L satisfying $GL = G_T$ and $LF_T = F$, then it must be such that $LX = FX$ for each object X in $\mathscr{C}_T$ and $Lf^\flat = \varepsilon_{FY} \circ Ff$ in $\mathscr{D}$ for each $f^\flat\colon X \to Y$ in $\mathscr{C}_T$. We additionally need to prove that $L\colon C_T \to \mathscr{D}$, characterized by $LX = X$ and $Lf^\flat = \varepsilon_Y \circ Ff$, is a functor satisfying $GL = G_T$ and $LF_T = F$:

1. For each X in $\mathscr{C}_T$, due to the fact that $id_X = (\eta_X)^\flat$ in $\mathscr{C}_T$, we have $L(id_X) = L((\eta_X)^\flat) = \varepsilon_{FX} \circ F\eta_X$. By [ML71, Ch. IV, §1, Theorem 1], we get $\varepsilon_{FX} \circ F\eta_X = id_{FX} = id_{LX}$. For each pair of morphisms $f^\flat\colon X \to Y$ and $g^\flat\colon Y \to Z$ in $\mathscr{C}_T$, by Kleisli composition, we obtain $L(g^\flat \circ f^\flat) = \varepsilon_{FZ} \circ FG\varepsilon_{FZ} \circ FGFg \circ Ff$. Since ε is natural, we have $\varepsilon_{FZ} \circ Fg \circ \varepsilon_{FY} \circ Ff$ which is $L(g^\flat) \circ L(f^\flat)$ in $\mathscr{D}$. Hence $L\colon \mathscr{C}_T \to \mathscr{D}$ is a functor.

2. For each object X in $\mathscr{C}_T$, $LX = FX$ in $\mathscr{D}$ and $GLX = GFX = TX = G_T X$ in $\mathscr{C}$. For each morphism $f^\flat\colon X \to Y$ in $\mathscr{C}_T$, $Lf^\flat = \varepsilon_{FY} \circ Ff$ in D by definition. Hence, $GLf^\flat = G\varepsilon_{FY} \circ GFf$. Similarly, Equation (5) gives $G_T f^\flat = G\varepsilon_{FY} \circ GFf$. We get $GLf^\flat = G_T f^\flat$ for each mapping $f^\flat$. Thus $GL = G_T$.

3. F_T is the identity on objects, thus $LF_T X = LX = FX$. For each morphism $f\colon X \to Y$ in $\mathscr{C}$, we have $F_T f = (\eta_Y \circ f)^\flat$ in $\mathscr{C}_T$, by definition. So that $LF_T f = L(\eta_Y \circ f)^\flat = \varepsilon_{FY} \circ F\eta_Y \circ Ff$. Due to ε and η being natural, we have $\varepsilon_{FY} \circ F\eta_Y = id_{FY}$ yielding $LF_T f = Ff$ for each mapping f. Therefore $LF_T = F$. □

A specialized version of the theorem is used by the author to model some formal logics in order to handle computational side effects in his thesis [Eki15]. An alternative definition of adjunctions is given in the following.

Definition 1.5. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ form an adjunction $F \dashv G\colon \mathscr{D} \to \mathscr{C}$ iff there exists *natural transformations* $\eta\colon Id_{\mathscr{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow Id_{\mathscr{D}}$ such that:

$$\varepsilon_{FX} \circ F\eta_X = id_{FX} \text{ for each } X \text{ in } \mathscr{C} \tag{6}$$

$$G\varepsilon_X \circ \eta_{GX} = id_{GX} \text{ for each } X \text{ in } \mathscr{D} \tag{7}$$

Lemma 1.6. *Definition 1.5* $\iff$ *Definition 1.1.*
See the proof in [Hen08, §3, Theorem 3.5].

2 Coq formalization

In a Coq implementation[1], we represent category theoretical objects such as functors, natural transformations, monads and adjunctions with data structures having single constructors and several fields, namely `classes`. This is no different than the approaches by Gross et al. [GCS14], Timany et al. [TJ16][2] and John Wiegley[3]. To our knowledge, none of them include the formalization of Mac Lane's comparison theorem. Also, our formalization makes use of proof irrelevance and functional extensionality axioms. These being said, let us start with the formalization of Definition 1.1:

```
Class HomAdjunction {C D: Category} (F: Functor D C) (G: Functor C D): Type ≜ mk_Homadj
  { ob: @Isomorphism (FunctorCategory (D^op × C) CoqCatT) (BiHomFunctorC F G) (BiHomFunctorD F G)  }.
```

An instance of the `HomAdjunction` class is defined as an isomorphism of bifunctors in the category of functors. In the above snippet, the notation `D^ op` denotes the dual of the category `D`, and `CoqCatT` represents the category of `Sets`. `BiHomFunctorC` implements the hom-functor $Hom_{\mathscr{C}}(X, GA)$ while `BiHomFunctorD` stands for the functor $Hom_{\mathscr{D}}(FX, A)$ in (1). On the other hand, Definition 1.5 looks like:

```
Class Adjunction {C D: Category} (F: Functor C D) (G: Functor D C): Type ≜ mk_Adj
  {  unit   : NaturalTransformation (@Id catC) (Compose_Functors F G);
     counit: NaturalTransformation (Compose_Functors G F) (@Id D);
     ob1    : ∀ a, (trans counit (fobj F a)) o fmap F (trans unit a) = @identity D (fobj F a);
     ob2    : ∀ a, (fmap G (trans counit a)) o trans unit (fobj G a) = @identity C (fobj G a)  }.
```

[1] https://github.com/ekiciburak/ComparisonTheorem-MacLane
[2] https://github.com/amintimany/Categories
[3] https://github.com/jwiegley/category-theory

where `unit` and `counit` correspond to η and ε as well as proof obligations `ob1` and `ob2` implement Equations (6) and (7) respectively. This means that to build an adjunction out of given categories and functors, one needs to provide two natural transformations satisfying the obligations. In the script, `fmap` is a field of the `Functor` type class that maps arrows while `fobj` is another field of the same class mapping objects of a domain category; `trans` is a filed of the `NaturalTransformation` class representing the component of the natural transformation at a given object. `Id` is the identity functor.

Formalizing in Coq Propositions 1.2, 1.3 and Theorem 1.4, we use `Adjunction` class instances instead of the ones of `HomAdjunction`. This is indeed not a problem thanks to Lemma 1.6. We have it certified in Coq:

```
Lemma adjEq1: ∀  (C D: Category) (F: Functor C D) (U: Functor D C), Adjunction F U     → HomAdjunction F U.

Lemma adjEq2: ∀  (C D: Category) (F: Functor C D) (U: Functor D C), HomAdjunction F U  → Adjunction F U.
```

We move on with the formalization of Proposition 1.2:

```
Theorem adj_mon  : ∀ {C D: Category} (F: Functor C D) (U: Functor D C), Adjunction F U  → Monad   C (Compose_Functors F U).

Theorem adj_comon: ∀ {C D: Category} (F: Functor C D) (U: Functor D C), Adjunction F U  → coMonad D (Compose_Functors U F).
```

See `Adjunctions.v` file for the proofs of so far stated theorems/lemmas. We implement Proposition 1.3 in three steps starting with the fact that every monad gives raise to a `Kleisli Category` whose objects are the ones of the base category `C` and morphisms are of the form $\mathtt{f}^\star\colon \mathtt{b} \to \mathtt{T}\,\mathtt{a}$ for each $\mathtt{f}\colon \mathtt{b} \to \mathtt{a}$ in `C`. Notice also that, nothing more than a design criteria, `@arrow C a b` implements a Coq type of maps defined from `b` to `a` in the category `C`:

```
Definition Kleisli_Category (C: Category) (T: Functor C C) (M: Monad C T): Category.
Proof. unshelve econstructor.
       - exact (@obj C).
       - intros a b. exact (@arrow C (fobj T a) b).
       ...
Defined.
```

Once obtaining this category, we can then claim that there is a special adjunction, namely `Kleisli adjunction`, between the base category `C` and the `Kleisli Category`. We implement the candidate adjoint functors as in Equations (4) and (5). Below, we only show the way they map objects and arrows respectively.

```
Definition LA {C D: Category} (F: Functor C D) (G: Functor D C) (T ≜ Compose_Functors F G) (M: Monad C T)
  (CT ≜ (Kleisli_Category C T M)): Functor C CT.
Proof. unshelve econstructor; simpl.
       - exact id.
       - intros a b f. exact (trans b o f).
       ...
Defined.

Definition RA {C D: Category} (F: Functor C D) (G: Functor D C) (T ≜ Compose_Functors F G) (M: Monad C T)
  (CT ≜ (Kleisli_Category C T M)): Functor CT C.
Proof. unshelve econstructor; simpl.
       - exact (fobj T).
       - intros a b g. exact (trans b o fmap T g).
       ...
Defined.
```

Above three definitions are implemented in the source `Monads.v`. We then prove that these candidate functors do actually form an adjunction:

```
Theorem mon_kladj: ∀ {C D: Category} (F: Functor C D) (G: Functor D C) (T ≜ Compose_Functors F G) (M: Monad C T)
  (FT ≜ LA F G M) (GT ≜ RA F G M), Adjunction FT GT.
```

Now, we can state Theorem 1.4 in Coq.

```
Theorem ComparisonMacLane: ∀ {C D: Category} (F: Functor C D) (G: Functor D C) (A1: Adjunction F G),
  let M  ≜ (@adj_mon C D F G A1) in let CT ≜ (Kleisli_Category C (Compose_Functors F G) M) in
  let FT ≜ (LA F G M) in let GT ≜ (RA F G M) in let A2 ≜ (mon_kladj F G M) in
    ∃ L: Functor CT D, Compose_Functors FT L = F ∧ Compose_Functors L G = GT.
```

Notice that proving this statement, we only get the existence of a comparison functor `L` satisfying the given properties but not that it is unique. Unicity proof is actually the work in progress. Also, we have formalized the dual of this theorem again leaving the unicity proof of comparison functor aside. See Coq proofs of two theorems above and the dual of the comparison theorem in the source `Adjunctions.v`.

3 Conclusion

We have formalized in Coq the comparison theorem without proving the uniqueness of comparison functor `L`. This uniqueness (also for the dual case) is the next property in our queue to formalize. Once getting these done, we plan to continue with implementing in Coq the proof of *Beck's theorem* which is a variant of comparison theorem where $\mathscr{C}_{\mathtt{T}}$ being the Eilenberg-Moore category of algebras of the monad `T`. The theorem claims that the comparison functor `L` is now an isomorphism.

3.1 Acknowledgements

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