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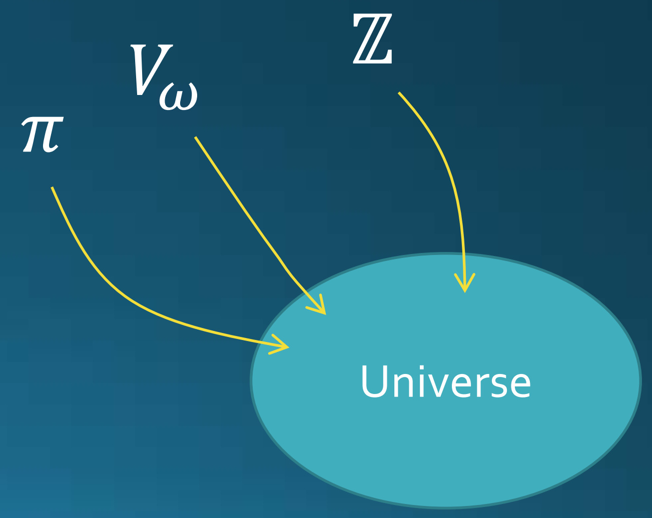
Models for Metamath

http://us2.metamath.org:88/model/model.pdf
http://us2.metamath.org:88/model/model.pptx

Metamathematics

- In the beginning, we have the universe of discourse
  - In set theory, these are the sets themselves, or the elements of a model



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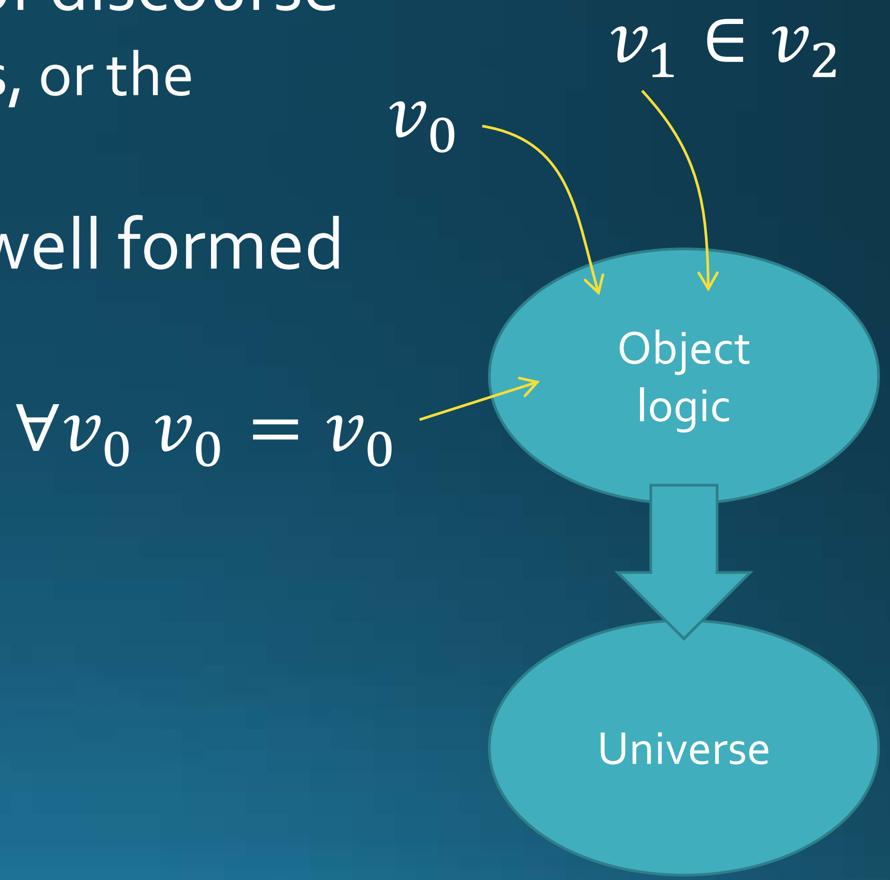
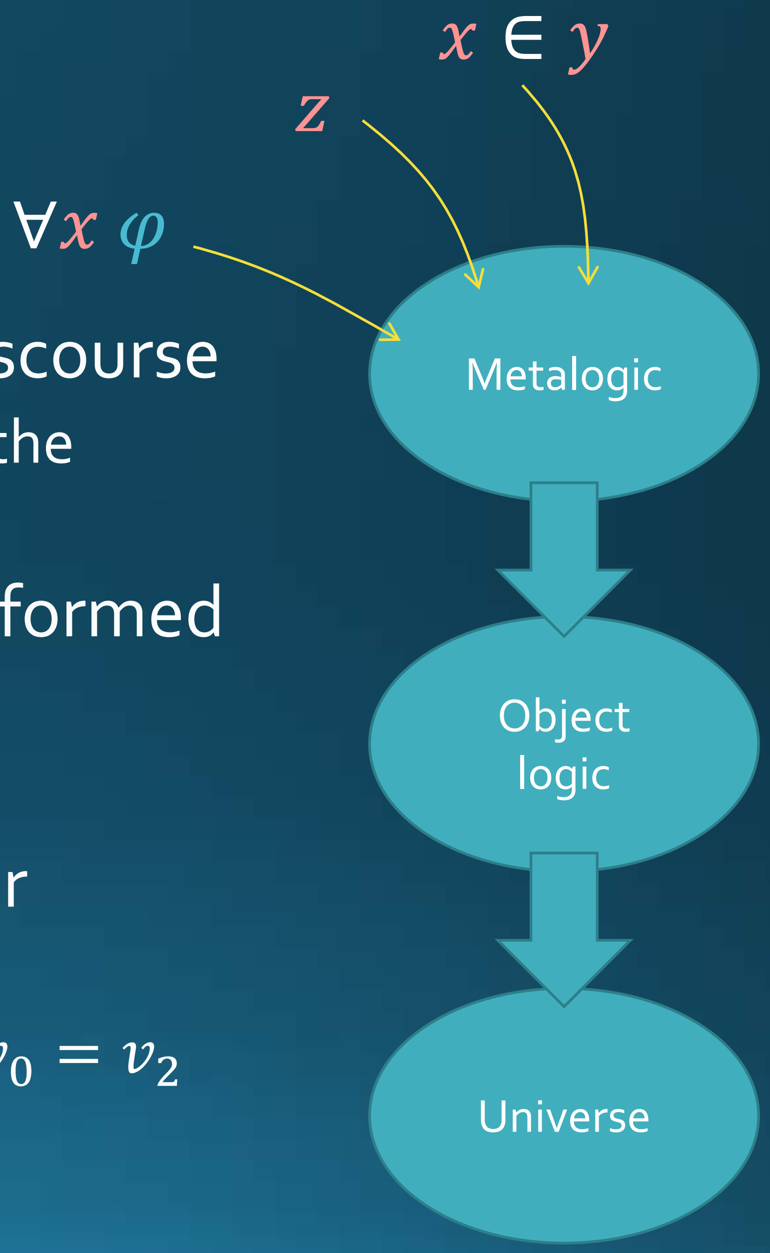
- The object logic contains variables and well formed formulas over the universe
  - All formulas use variables, $\forall, =, \in$



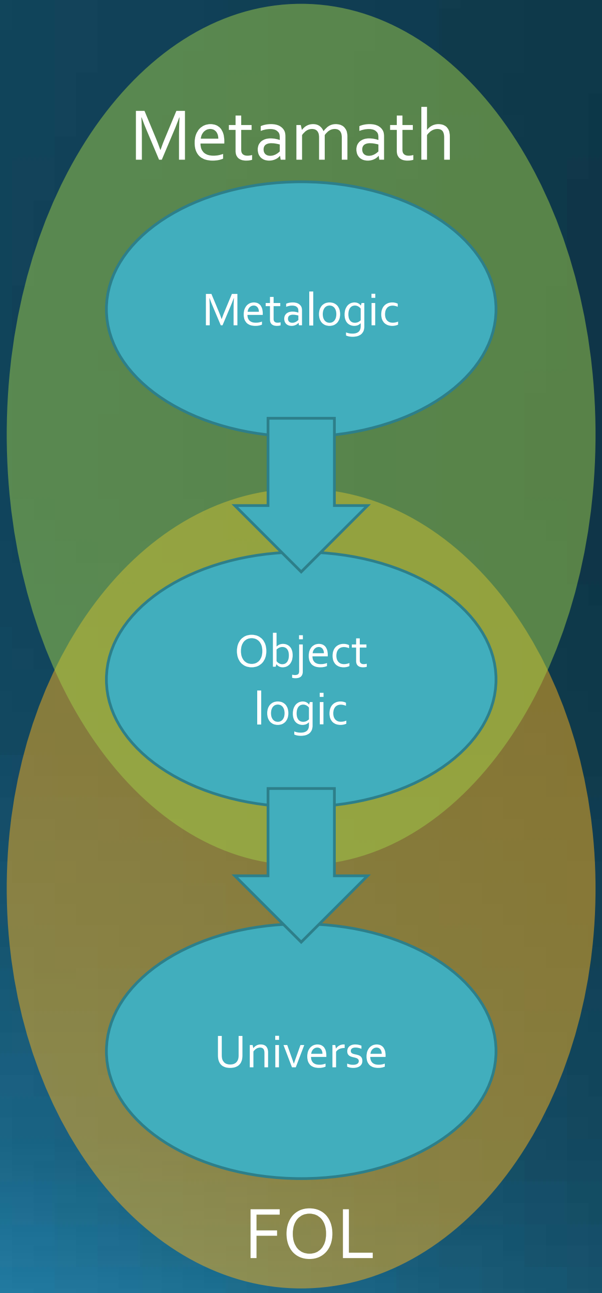
Metamathematics

- In the beginning, we have the universe of discourse
  - In set theory, these are the sets themselves, or the elements of a model
- The object logic contains variables and well formed formulas over the universe
  - All formulas use variables, $\forall, =, \in$
- The metalogic uses variables that range over variables of the object logic
  - So x could be v_0, v_1, etc. while φ could be $\forall v_0\ v_0 = v_2$



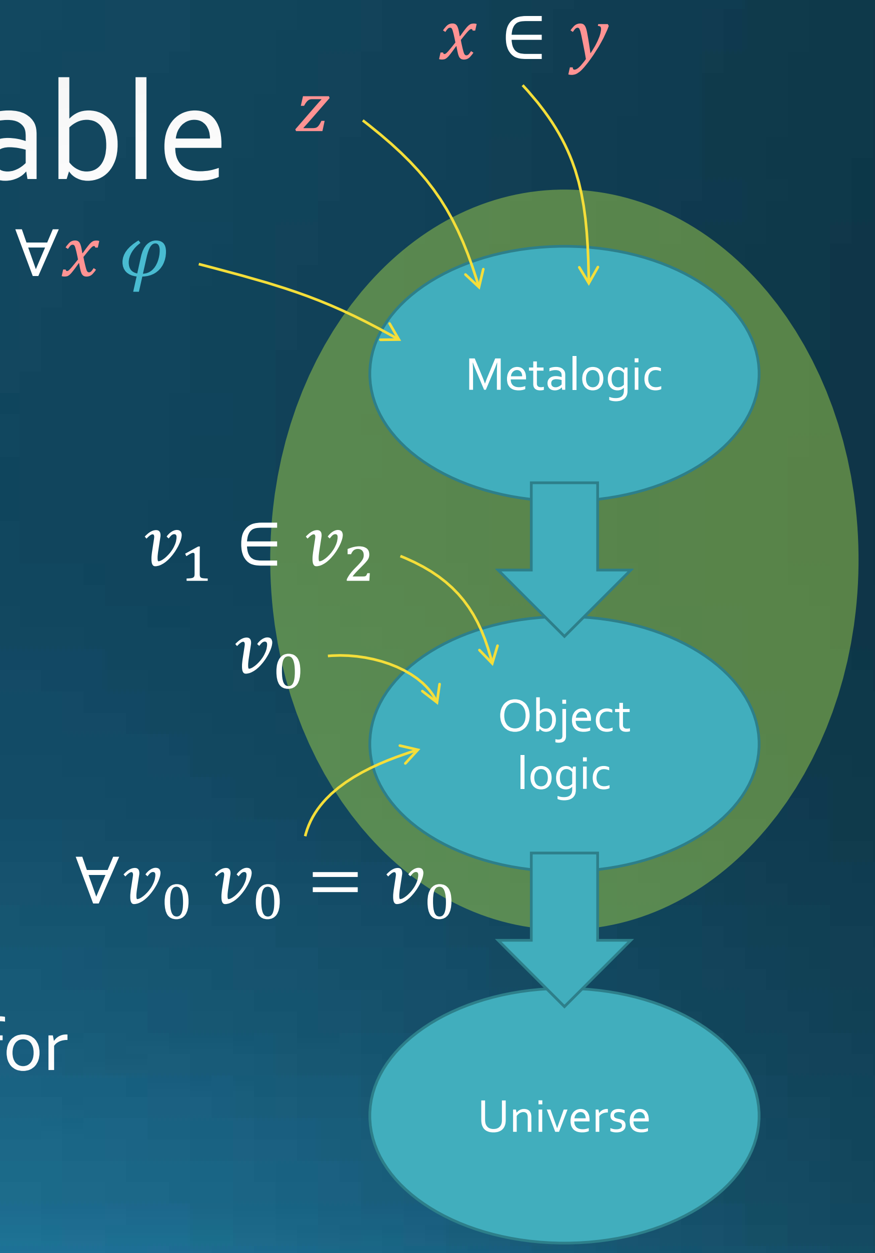
Metamathematics

- FOL handles the relationship between the object logic and the universe
- Metamath handles the relationship between the metalogic and the object logic
  - Metamath does not model the universe!
  - How to axiomatize?



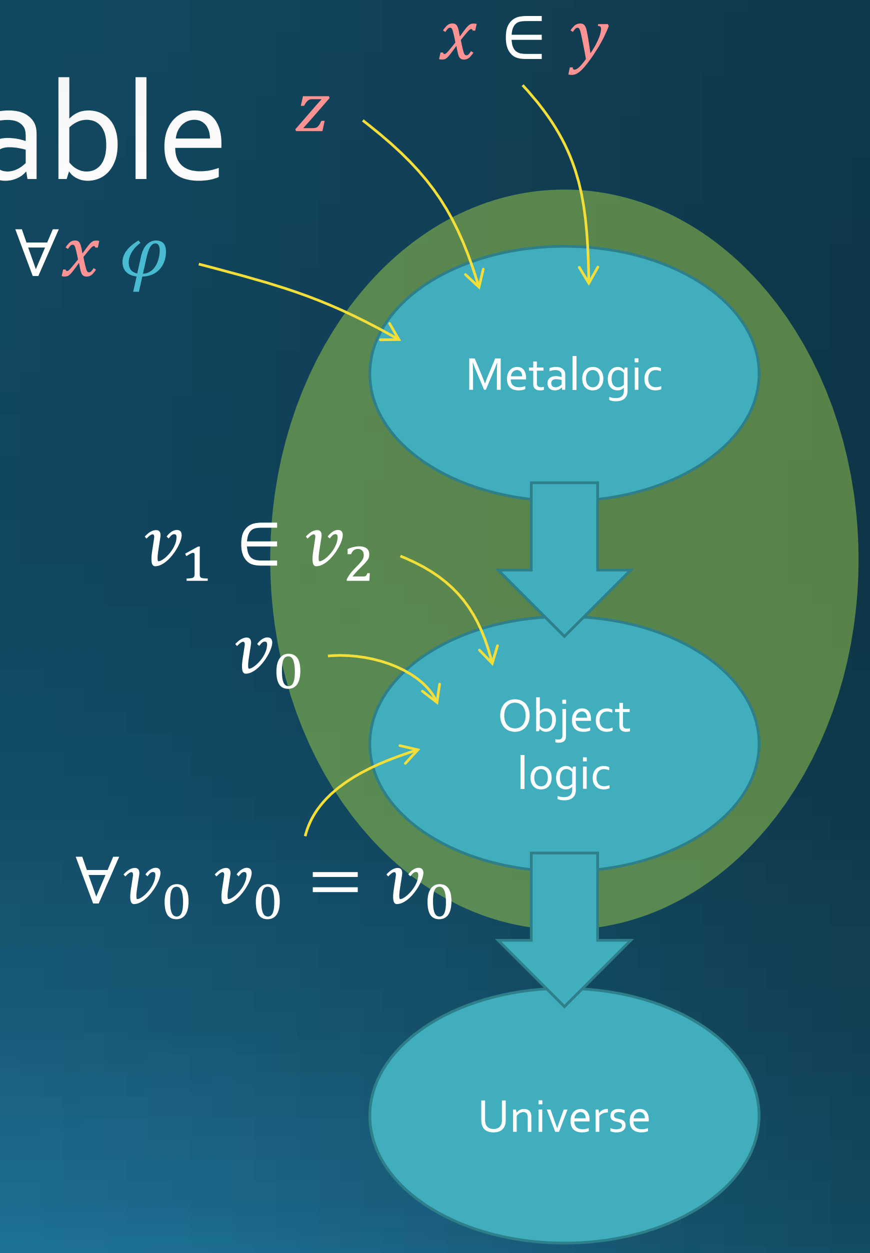
Properties of a metavariable

- Variables substitute for expressions of the same type in the object logic
  - i.e. $x \mapsto v_0$, $\varphi \mapsto v_0 = v_0$
- Different simultaneous substitutions may contain the same variable
  - $x \mapsto v_0$, $y \mapsto v_0$ is *legal*
  - $x \mapsto v_0$, $\varphi \mapsto v_0 = v_0$ has a shared substitution variable v_0 between x, φ
- There are enough dummy variables, i.e. v_k for arbitrarily large k



Properties of a metavariable

- True or false: $\forall x\, x = y$
  - It depends on the values of x, y
  - If $x = y$ as variables, i.e. $x, y \mapsto v_n$, then $\forall v_n\, v_n = v_n$ is true
  - If $x \neq y$, i.e. $x \mapsto v_m, y \mapsto v_n$ for $m \neq n$, then $\forall v_m\, v_m = v_n$ is false
- Solution: disjoint variable provisos
  - $\vdash \neg \forall x\, x = y$, provided x, y are disjoint
  - $\vdash (\varphi \rightarrow \forall x\, \varphi)$, provided x, φ are disjoint
  - These are *not* bound variable conditions



Metamath

- Metavariables all the way (no direct reference to object variables)
- Each metavariable has a type, e.g. set x, wff φ means x is a metavariable of type set (set variables) and φ is a wff metavariable
- If a theorem $\vdash \forall x(\varphi \to \varphi)$ is proven for metavariables x, φ we can substitute any metavariable expression provably of that type for each metavariable
  - i.e. $\vdash \forall x(\varphi \to \varphi)$ can substitute $x \mapsto y, \varphi \mapsto x = y$ to get $\vdash \forall y(x = y \to x = y)$
  - This a basic (built-in) axiom of Metamath which is justified since the original theorem represents a theorem scheme of formulas of the object logic that contains all instances of the substitution's scheme

Metamath

- Disjoint variable provisos distribute over substitution
  - $\vdash (\varphi \rightarrow \forall x\, \varphi)$ provided x, φ are disjoint
  - with $x \mapsto y$, $\varphi \mapsto x = z$
  - gives $\vdash (x = z \rightarrow \forall y\, x = z)$ provided x, y and y, z are disjoint
- A variable is not disjoint with itself
  - Above theorem with substitution $x \mapsto y$, $\varphi \mapsto x = y$ gives $\vdash (x = x \rightarrow \forall y\, x = x)$ provided x, y and y, y are disjoint which is impossible, so this is invalid

Grammars and trees

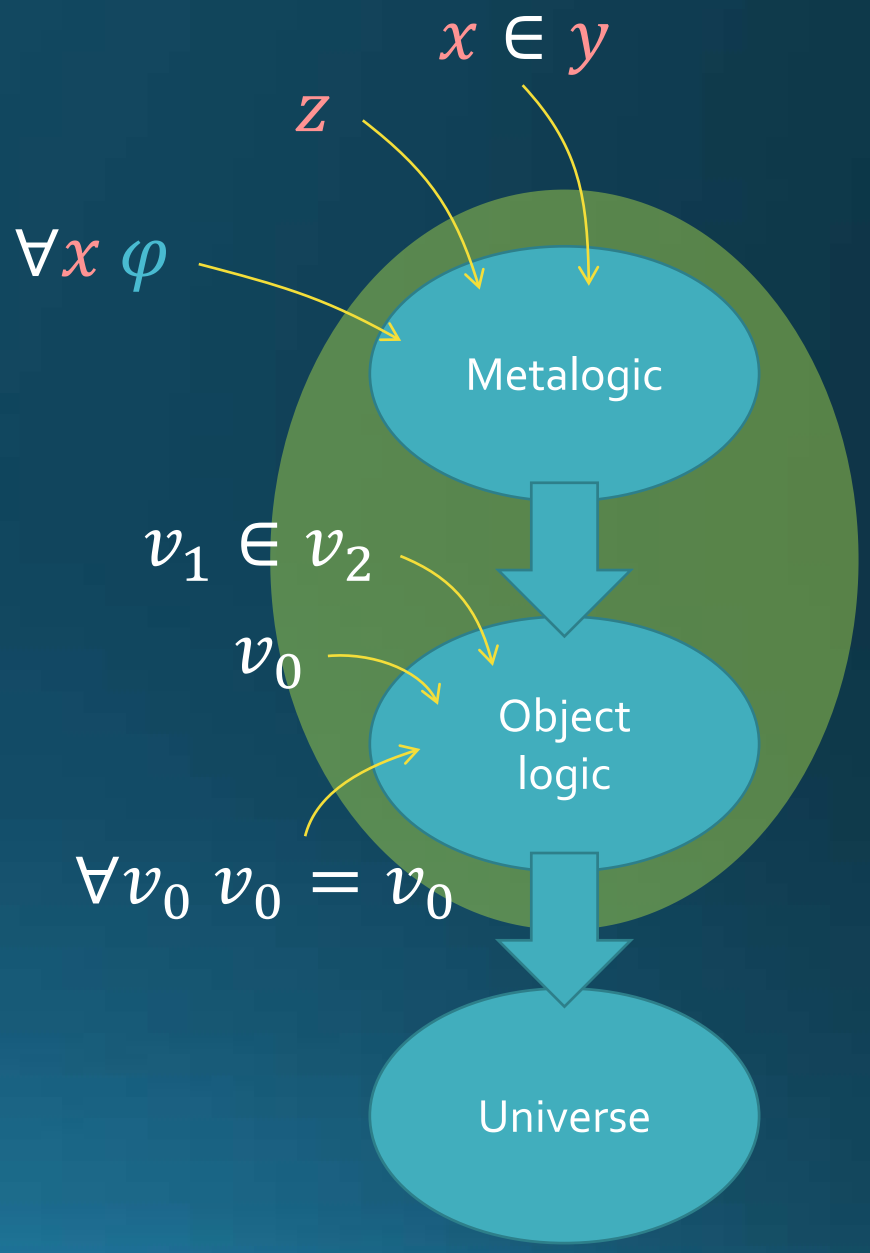
- For speed/historical reasons, expressions are strings, not trees
  - AKA Metamath is a string rewriting system
  - Cute example: Hofstadter’s MIU system is valid Metamath
  - ...but I don’t like strings, they are hard to reason with
  - Under what conditions is Metamath isomorphic to a tree-based substitution system?
- Solution: Detect the grammar

Grammars and trees

- Syntax axioms are axioms whose consequent is a variable type
  - Ex: wff $x = y$
    - Not a variable type: $\vdash x = x$
  - Syntax axioms are not allowed to reuse variables, and are not allowed to have disjoint variable provisos or (explicit) hypotheses
  - Result: a context free grammar
- An unambiguous formal system is a Metamath database in which the resulting grammar is unambiguous
  - Equivalently: each string expression with a variable type has at most one proof
  - Result: one-to-one correspondence to trees, and we can henceforth pretend that this is how Metamath was defined

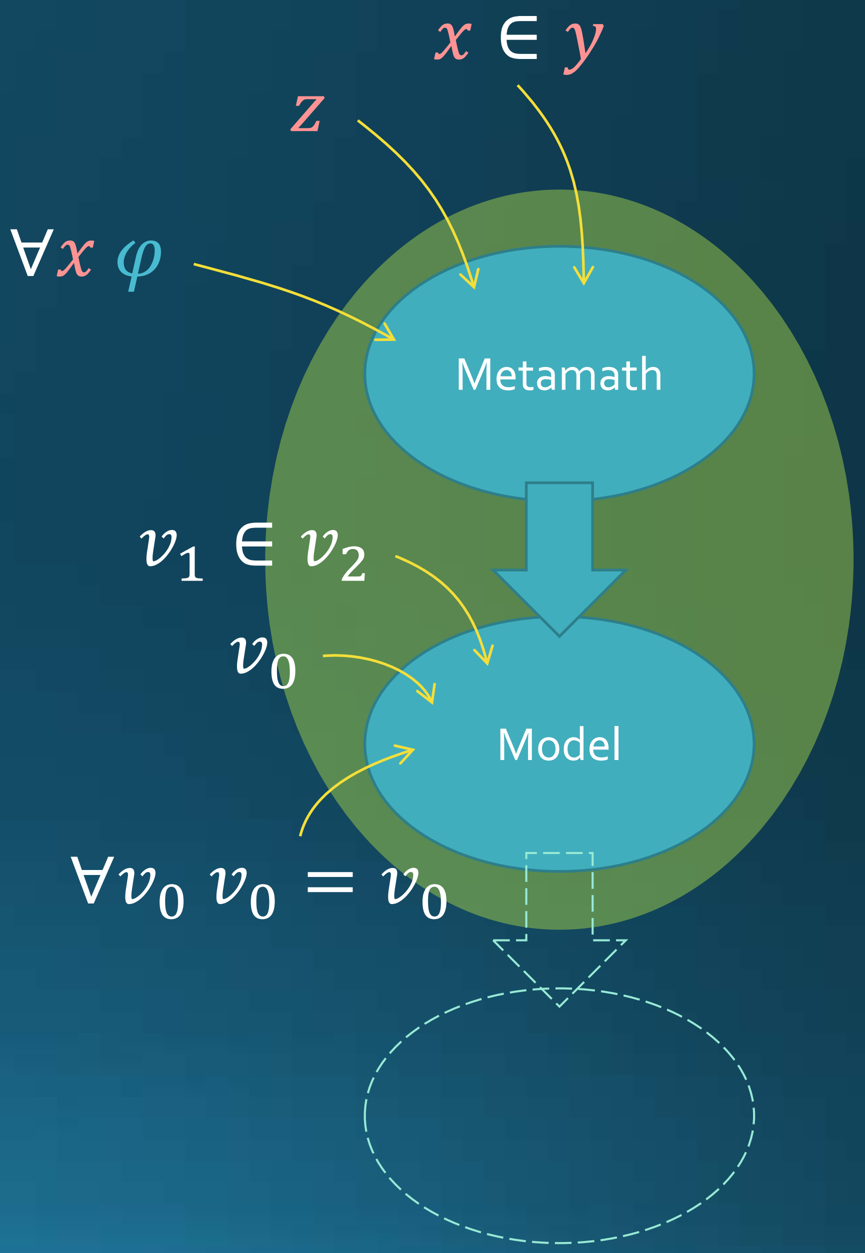
Models

- Return to the original picture



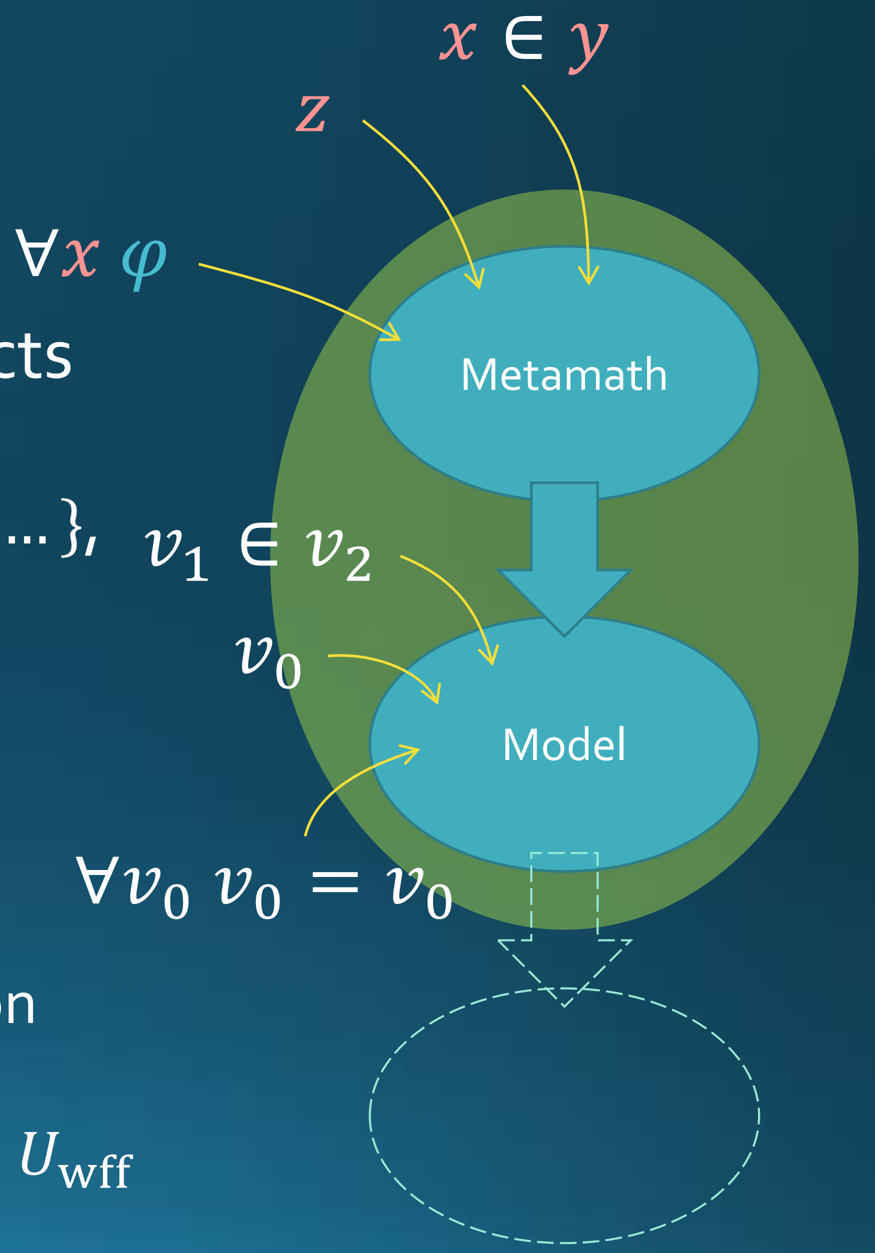
Models

- Return to the original picture
- A model is a space that acts like the object logic over a universe
  - No requirement that the universe exists
  - It only has to satisfy the properties expected by Metamath
- Models for string systems: read the paper



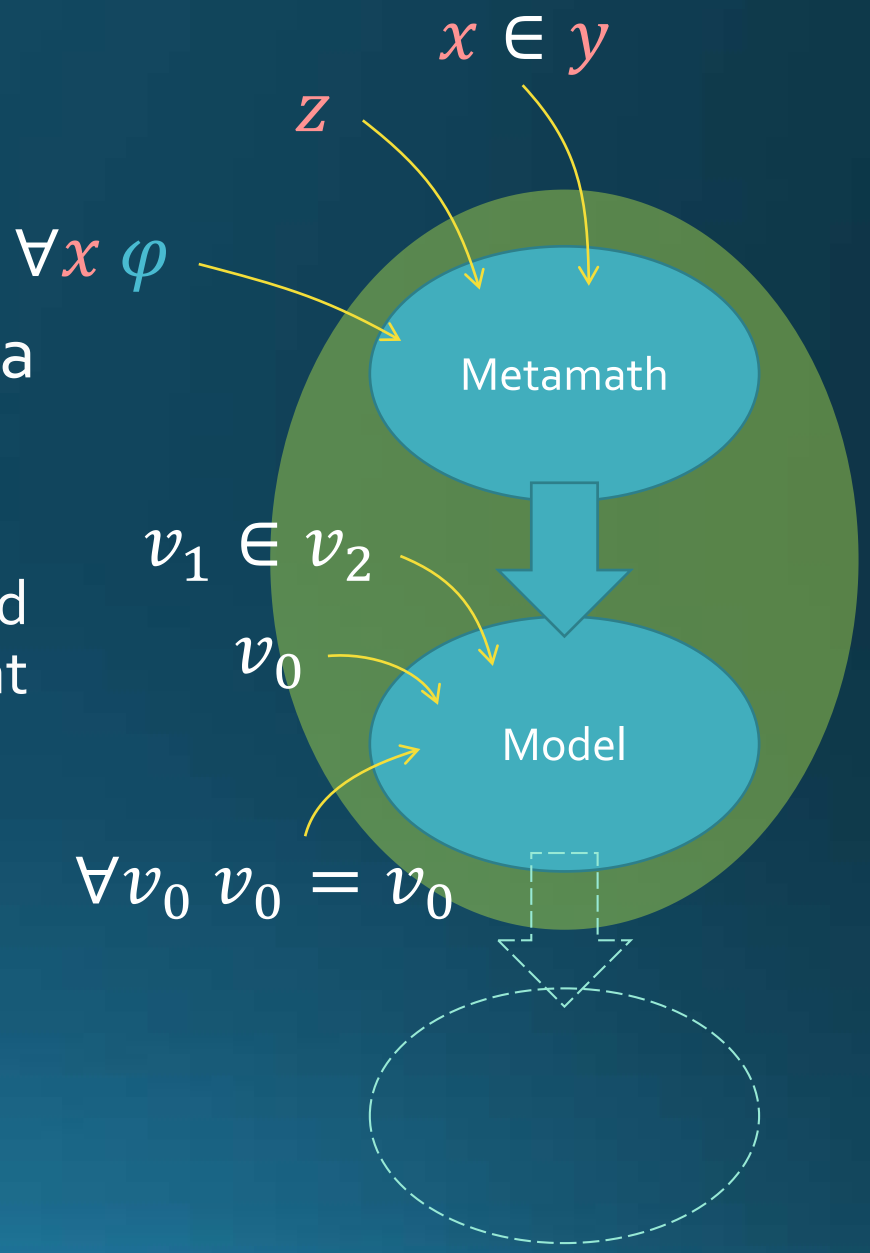
Models

- For each type, there is a universe U_c of objects of that type
  - i.e. U_{set} is the collection of set variables $\{v_0, v_1, \ldots\}$, U_{wff} is the set of relations on M depending on finitely many of the v_k
    - ...maybe
- Syntax axioms are well-typed functions on the universe
  - i.e. implication, wff $(\varphi \to \psi)$ becomes a function $(U_{\mathrm{wff}}, U_{\mathrm{wff}}) \to U_{\mathrm{wff}}$
  - Forall is wff $\forall x\ \varphi$ which has type $(U_{\mathrm{set}}, U_{\mathrm{wff}}) \to U_{\mathrm{wff}}$
    - Not $(M \to U_{\mathrm{wff}}) \to U_{\mathrm{wff}}$!



Models

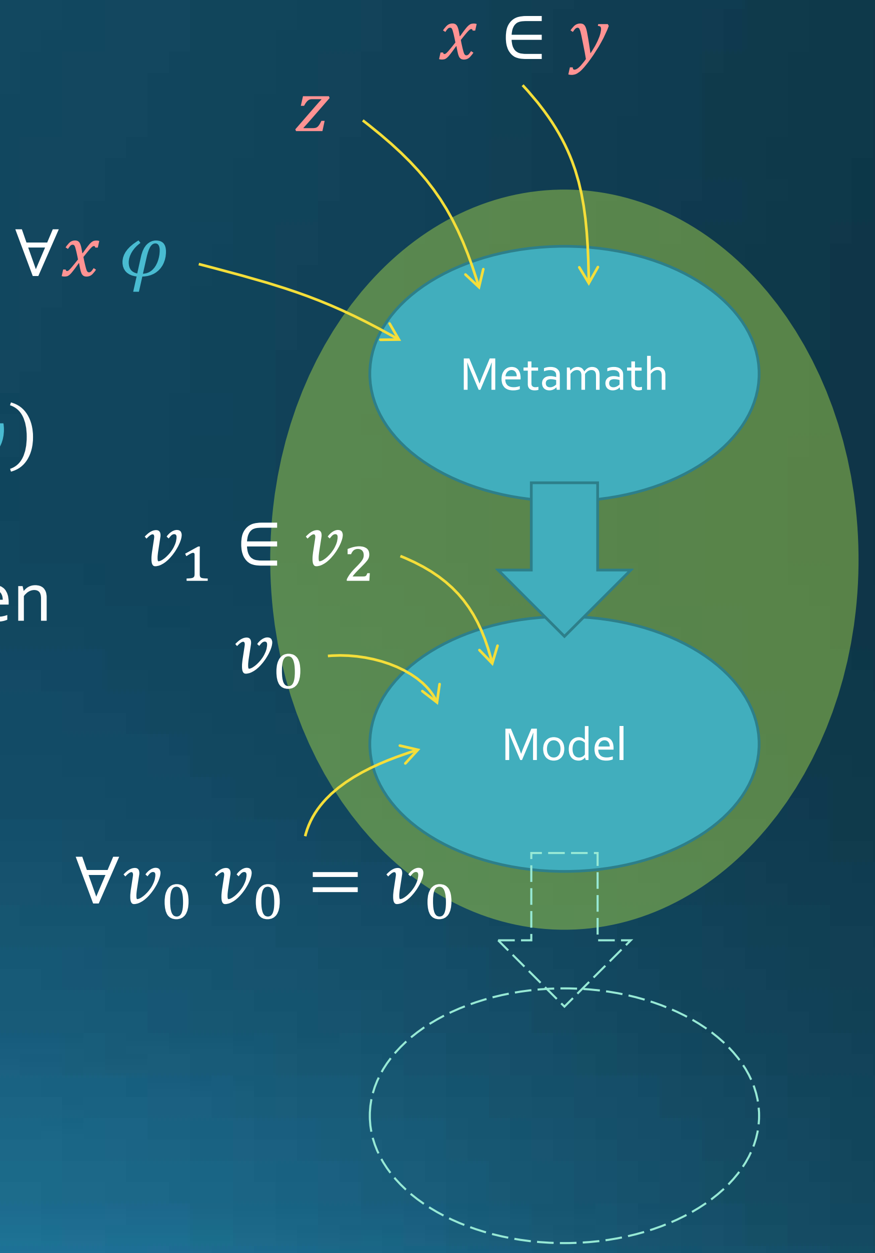
- Disjoint variable conditions are modeled by a relation $v \# w$ that holds between elements of the universe
  - For elements of U_{set}, this is $v \# w \leftrightarrow v \neq w$, and for $v \# f$ with $v \in U_{\text{set}}$, $f \in U_{\text{wff}}$ this means that f is constant with respect to the variable v
- A special subset $U_{\vdash} \subseteq U_{\text{wff}}$ gives the set of model elements that are considered to be true in the model
  - This is just the singleton of the always true relation



Models

A theorem like
x, ψ disjoint & $\vdash (\varphi \to \psi) \Rightarrow \vdash (\exists x\, \varphi \to \psi)$
is valid whenever for all $\bar{x} \in U_{\text{set}}$ and all $\bar{\varphi}, \bar{\psi} \in U_{\text{wff}}$, if $\bar{x} \mathbin{\#} \bar{\psi}$ and $\text{imp}(\bar{\varphi}, \bar{\psi}) \in U_{\vdash}$, then $\text{imp}(\text{ex}(v, \bar{\varphi}), \bar{\psi}) \in U_{\vdash}$

- This is an FOL statement!



It all works out

- Soundness
- There is a model for set.mm, assuming that ZFC has a model
  - Converse?
- Godel completeness
  - A theorem is true iff it is true in every model
  - What about disjoint variables and hypotheses?
- Independence proofs of set.mm axioms
- Conservative extensions of a theory, what happens to the model
- The category of models and model homomorphisms

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What's the point

- Faithful representation of *full* Metamath in FOL
  - Metamath → TPTP
  - Metamath → MMT
  - Metamath → HOL/DTT
  - Downside: not a nice embedding
- Better faithful representation of unambiguous Metamath in FOL
  - set.mm → TPTP, MMT, etc.
  - Still not as good as we'd like
- Future work: *Intended* embedding of Metamath in FOL
  - So that forall is a binder, not a binary function
  - Can be dealt with in this framework

Questions