A formalisation of metric spaces in HOL Light

Marco Maggesi** [1]

DiMaI - Dipartimento di Matematica e Informatica "U. Dini"
Università degli Studi Firenze, Italy
`http://www.math.unifi.it/~maggesi/`

Abstract. We present a computer formalisation of metric spaces in the HOL Light theorem prover. Basic results of the theory of complete metric spaces are proved. A simple decision procedure for the theory of metric space is implemented.

Keywords: Metric spaces, Higher-Order Logic, Formalisation of mathematics

1 Introduction

Metric spaces constitute an unavoidable concept in several mathematical fields like geometry, topology, and analysis. We introduce the definition of metric space in HOL Light and we prove some classical results and applications.[1]

1.1 Background

Several important results of the theory of metric spaces have been already formalised by Harrison in HOL Light in the context of euclidean geometry [Har05]. The main point of the present work is to setup in HOL Light a general theory of *abstract* metric space in which the theorems of metric geometry can be stated and exploited in its full generality, as it is required for their application in various fields, like in algebra or in functional analysis.

Part of the code presented in this paper is based on a previous work of Claudia Carapelle who developed the definition of metric spaces in HOL Light for her master thesis in Mathematics at the University of Florence [Car11].

1.2 Related works

Being such a fundamental tool, the theory of metric spaces, up to a variable degree of details and generality, appears in several formalisation efforts. We limit ourselves to cite only two very recent achievements.

** The author was supported by MIUR and GNSAGA-INdAM.

[1] Our code has been included in the HOL Light distribution and is also available from its original repository at `https://bitbucket.org/maggesi/metric/`.

Immler and Hölzl use metric spaces [IH12] to give a proof of the Picard-Lindelöf theorem with a constructive approach, thus providing a numerical approximation method for the solution of ordinary differential equations.

Another work with a strong emphasis on constructivity has been carried on in Coq by Makarov and Spitters in [MS13] where they give an intuitionistic proof of the Picard-Lindelöf theorem.

2 Metric spaces in Higher-Order Logic

In mainstream mathematics, with the language of sets, a metric space M is defined as a pair (X, d) where X is a set of *points* and d is a *metric* on X, that is a function $d \colon X \times X \to \mathbb{R}$ such that the following holds: for any three points x, y, z of X

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, y) + d(y, z)$.

Our definition in Higher-Order Logic mimic the set-theoretical one:[2]

```
let is_metric_space = new_definition
  `is_metric_space (s,d) <=>
   (!x y:X. x IN s /\ y IN s ==> &0 <= d(x,y)) /\
   (!x y. x IN s /\ y IN s ==> (d(x,y) = &0 <=> x = y)) /\
   (!x y. x IN s /\ y IN s ==> d(x,y) = d(y,x)) /\
   (!x y z. x IN s /\ y IN s /\ z IN s
            ==> d(x,z) <= d(x,y) + d(y,z))`;;
```

Notice that for the elements outside the 'set of points' s, nothing can be deduced about the behaviour of the metric d.

In the `Multivariate_Analysis` library of Isabelle/HOL, a slight different definition is adopted, where the set of points is made to coincide with the whole ambient type `` `:X` ``. This is especially convenient in Isabelle, since it allows to exploit the mechanism of axiomatic classes provided by the system. However, our definition is more general, since, in particular, it allows to formalise in a natural way families of metric spaces, like the L^p spaces used in functional analysis.

3 Complete metric spaces

Most topological and metric phenomena are best studied and understood trough the behaviour of *sequences*. We prove various metric and topological results about sequences in metric spaces.

[2] The symbol `` `&` `` is the embedding $\mathbb{N} \to \mathbb{R}$. Other syntax elements should be clear from the context.

In particular, a crucial notion in metric space is the one of *Cauchy sequence*, which means that the distance between its elements become arbitrary small after a certain index. It is an easy but fundamental fact that every convergent sequence is Cauchy sequence. A metric space is said to be *complete* if the converse it is also true.

We formalise the basics of the theory of complete metric spaces. In particular, we show the completeness of certain metric spaces and we prove certain classical results, like the Banach Fixed-point Theorem and the Baire Category Theorem.

As an example, we illustrate the statement of the Baire Category Theorem:

```
METRIC_BAIRE_CATEGORY
|- !m g.
      mcomplete m /\
      COUNTABLE g /\
      (!t. t IN g ==> open_in (mtopology m) t /\
                      mtopology m closure_of t = mspace m)
      ==> mtopology m closure_of INTERS g = mspace m
```

In the above statement, we have a complete metric space `‘m‘` (whose associated set of points is `‘mspace m‘` and whose associated topology is `‘mtopology m‘`) and a countable family of dense open sets `‘g‘`. The thesis is that the intersection of the family `‘g‘` is dense in `‘m‘`.

One key example in this work is the space of continuous bounded functions, which is endowed with a structure of metric space with the L^∞-metric: given a topological space X and a metric space M, the distance in L^∞ between two bounded continuous functions $f, g\colon X \longrightarrow M$ is defined by

$$d_\infty(f,g) = \sup_{x\in X} d_M(f(x), g(x)).$$

Such space (denoted `‘cfunspace top m‘`) is complete when the target space M is complete:

```
MCOMPLETE_CFUNSPACE
  |- !top m. mcomplete m ==> mcomplete (cfunspace top m)
```

We will use this fact in the proof of the Picard-Lindelöf theorem (see below).

There is one technical point which is worth to mention about the formalisation of function spaces. We want to work with *partial* function, i.e., functions whose intended domain is a subset of their domain type. Hence, to preserve the property of indiscernibility (the fact that two functions are the same if their distance is zero), we are forced to setup a mechanism for 'truncating' a function to its 'domain of definition'. For the sake of brevity, we won't delve into the technical details in this extended abstract.

To give a significant application of our formalisation, we prove one classical corollary of the Banach fixed-point theorem in analysis: the Picard-Lindelöf theorem about the existence and uniqueness of the solution of ordinary differential equations. Beside being an interesting achievement in its own, we present this further development as a testbed for the soundness of our approach and the applicability of our constructions.

4 A decision procedure for the theory of metric spaces

Solovay, Arthan and Harrison provide in [SAH09] a collection of decidability results for various algebraic and geometric theories, including metric spaces. On that basis we have implemented a decision procedure, `METRIC_ARITH`, which automate certain proofs such as reasoning with the triangular law.

We give an example of its usage: we will show that the distance between the centers x, y of two balls $B(y, s)$ and $B(z, t)$ containing a common ball $B(x, r)$ is lower than the sum of their radius $s + t$. The complete proof script is the following:

```
g `!m x y z:Y r s t.
     x IN mspace m /\ y IN mspace m /\ z IN mspace m /\ &0 < r /\
     mball m (x,r) SUBSET mball m (y,s) /\
     mball m (x,r) SUBSET mball m (z,t)
     ==> mdist m (y,z) <= s + t`;;
e (SIMP_TAC[SUBSET; IN_MBALL]);;
e METRIC_ARITH_TAC;;
top_thm();;
```

5 Conclusions

Metric spaces are an indispensable tool in modern mathematics. We introduced a definition of metric space which allow to state and prove theorems about metric geometry in its full generality. We implemented a simple decision procedure, we proved some notable results about complete metric spaces and we gave some basic applications to functional analysis and ordinary differential equations. Our hope is that the present work can lay a foundation for the theory of metric spaces in the HOL Light theorem prover.

References

Car11. Claudia Carapelle. La formalizzazione degli spazi metrici e teorema di punto fisso nella logica di ordine superiore. Master's thesis, Corso di Laurea Specialistica in Matematica, Università degli Studi di Firenze, 2011.

Har05. John Harrison. A HOL theory of Euclidean space. In Joe Hurd and Tom Melham, editors, *Theorem Proving in Higher Order Logics, 18th International Conference, TPHOLs 2005*, volume 3603 of *Lecture Notes in Computer Science*, pages 114–129, Oxford, UK, August 2005. Springer-Verlag.

IH12. Fabian Immler and Johannes Hölzl. Numerical Analysis of Ordinary Differential Equations in Isabelle/HOL. In Lennart Beringer and Amy Felty, editors, *Interactive Theorem Proving*, volume 7406 of *Lecture Notes in Computer Science*, pages 377–392. Springer Berlin Heidelberg, 2012.

MS13. Evgeny Makarov and Bas Spitters. The Picard Algorithm for Ordinary Differential equations in Coq. In Sandrine Blazy, Christine Paulin-Mohring, and David Pichardie, editors, *Interactive Theorem Proving*, volume 7998 of *Lecture Notes in Computer Science*, pages 463–468. Springer Berlin Heidelberg, 2013.

SAH09. R. M. Solovay, R. D. Arthan, and J. Harrison. Some new results on decidability for elementary algebra and geometry. *ArXiv e-prints*, April 2009.