Developments, Libraries and Automated Theorem Provers

Chad E. Brown

Czech Technical University in Prague

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Outline

Introduction

2009- SATALLAX

Higher Order Automated Theorem Prover

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Proof Checker, Higher Order Tarski-Grothendieck Set Theory

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mid 1990s
Otter (McCune)? Foundation? NBG (Quaife)?

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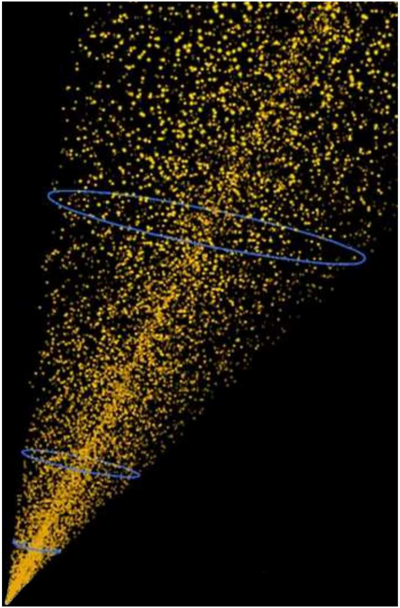
- *"Everything is a set."* E.g., pairs are sets: $(x, y) = \{\{x\}, \{x, y\}\}$
- A model of, e.g., Tarski-Grothendieck is a kind of "monster model" of mathematics.

Iterative Conception of the Universe



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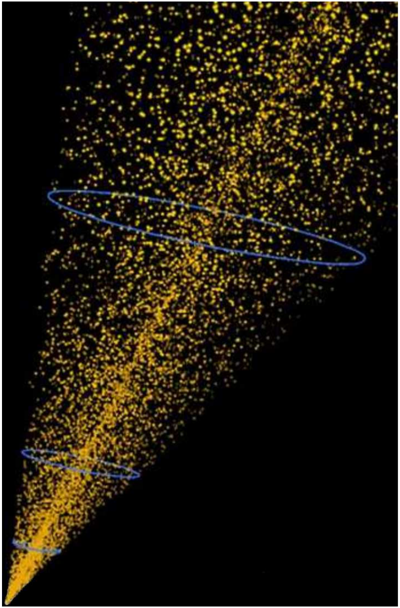

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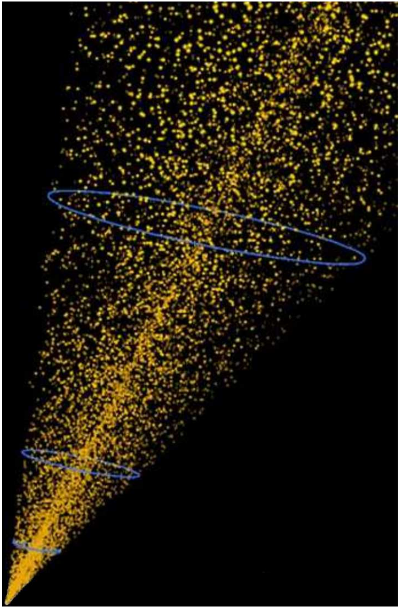

Hereditarily Finite Sets

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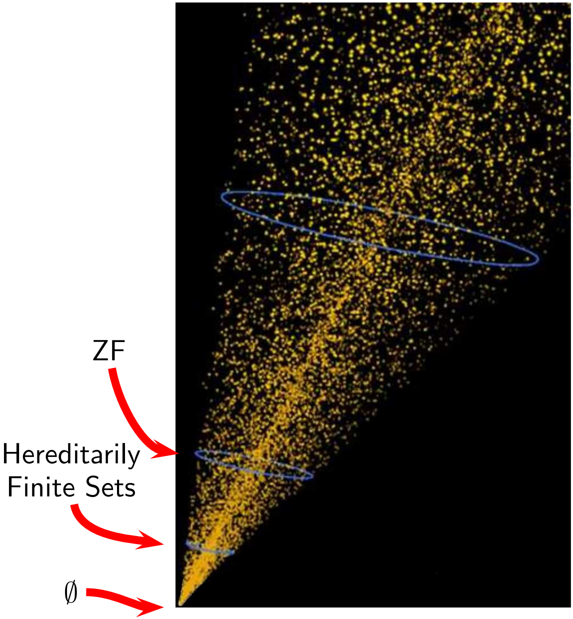

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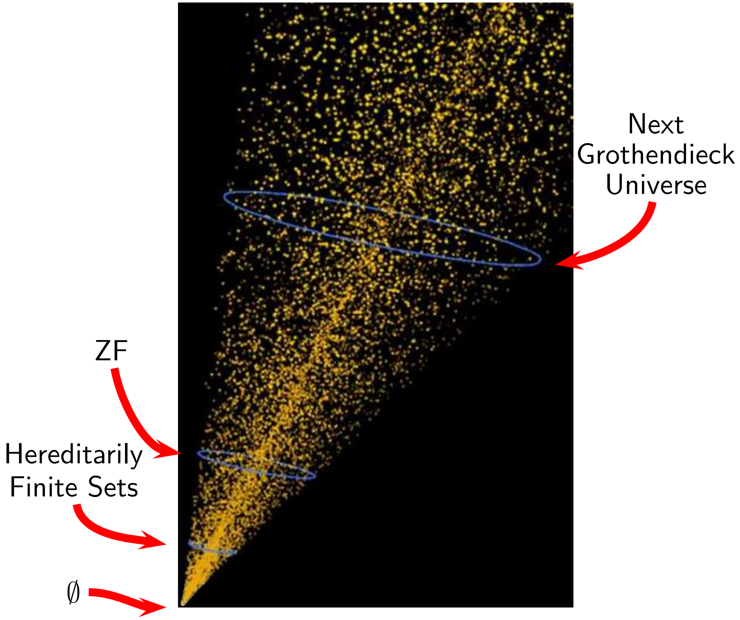

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- $\iota \to \cdots \to \iota \to o$: predicates/classes
- $\iota \to \cdots \to \iota \to \iota$: metafunctions

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- Question: Is this already in the MML?
  (Mizar Mathematical Library)

Local Developments vs a Global Library

Library

Developments
(Documents/Articles/Theories)
Local

Local Developments vs a Global Library

Global
(Theorems, Definitions)
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Local Developments vs a Global Library

Global
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search

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- Satallax extended to search for "similar" theorems to a conjecture
- Search for all theorems of the form

$$\forall Y : \mathtt{set}. * (*(Y)) = Y$$

Searching the MML for U(I(Y))=Y

```
include('allofmizar-preface-types-axioms').

thf(invset,conjecture,
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~/satallax/satallax.native -similar invset
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Searching for similar formulas.
Number of conclusion variants: 3
Total partial matches: 2376
t2_jgraph_1 similarity measure 0
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t5_pcomps_1 similarity measure 0
t19_cohsp_1 similarity measure 0
t25_zfmisc_1 similarity measure 0
t19_yellow_1 similarity measure 0
t55_modelc_3 similarity measure 0
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t2_cohsp_1 similarity measure 0
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Total of 13 of measure 0
t37_ordinal1 similarity measure 1
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None of these 13 correspond to $U(I(Y)) = Y$ for my U and I.

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$U(I(Y)) = Y$ can be proven by $\in$-induction.
$\in$-induction: For every property P of sets,

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- Does it easily follow from the MML?

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Finite Sets, No Help

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s1_ordinal2 similarity measure 4
s2_hilbert2 similarity measure 4
s3_cgames_1 similarity measure 4
s2_nat_2 similarity measure 4
s1_zf_lang similarity measure 4
s1_chord similarity measure 4
Total of 26 of measure 4
s4_int_1 similarity measure 5
```

No Exact Matches, 92 Partial Matches

Finite Sets, No Help

Natural Number Induction, No Help

Searching the MML for $\in$-induction

```
Read it, initial_branch has 135002 elts
Searching for similar formulas.
Number of conclusion variants: 2
Total partial matches: 92
s2_finset_1 similarity measure 3
s17_ordinal2 similarity measure 3
Total of 2 of measure 3
s1_card_1 similarity measure 4
s1_uniroots similarity measure 4
s1_fib_num2 similarity measure 4
s3_finseq_1 similarity measure 4
s2_fib_num2 similarity measure 4
s2_pre_poly similarity measure 4
s2_nat_d similarity measure 4
s5_funct_7 similarity measure 4
s3_afinsq_1 similarity measure 4
s1_card_fil similarity measure 4
s1_fib_num similarity measure 4
s1_zf_lang1 similarity measure 4
s1_modal_1 similarity measure 4
s2_ordinal1 similarity measure 4
s4_nat_1 similarity measure 4
s1_nat_1 similarity measure 4
s1_modelc_2 similarity measure 4
s2_nat_1 similarity measure 4
s10_nat_1 similarity measure 4
s1_glib_000 similarity measure 4
s1_ordinal2 similarity measure 4
s2_hilbert2 similarity measure 4
s3_cgames_1 similarity measure 4
s2_nat_2 similarity measure 4
s1_zf_lang similarity measure 4
s1_chord similarity measure 4
Total of 26 of measure 4
s4_int_1 similarity measure 5
```

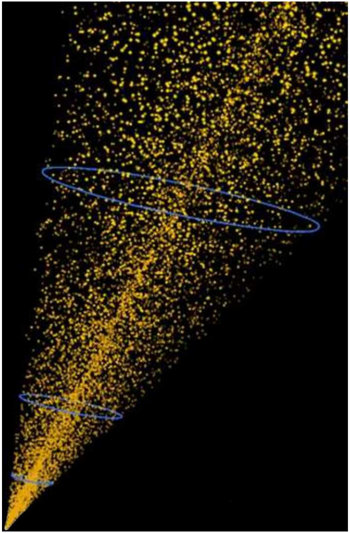
No Exact Matches, 92 Partial Matches

Finite Sets, No Help

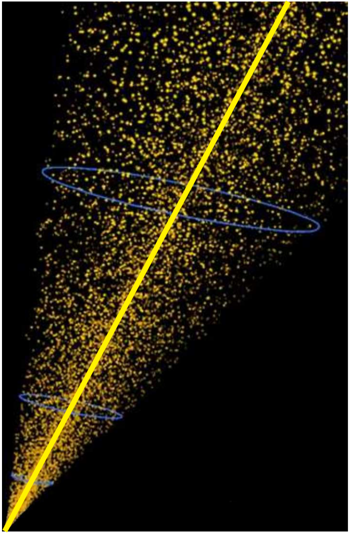
Natural Number Induction, No Help

Transfinite Induction!

Transfinite Induction

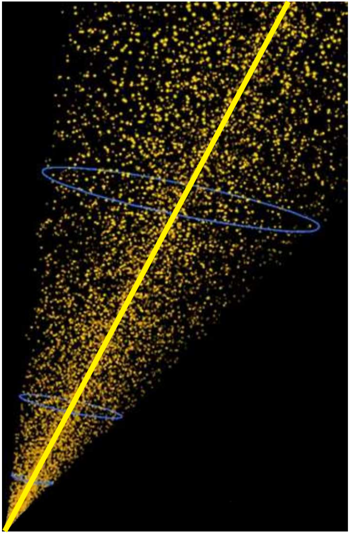


Transfinite Induction



Ordinals

Transfinite Induction

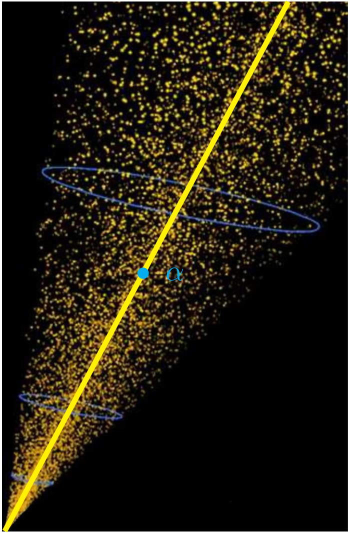


Ordinals
$P[Ordinal]$

$\rightarrow$

$\forall \alpha . P[\alpha]$

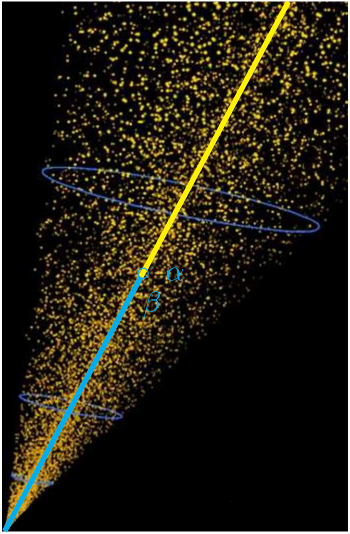
Transfinite Induction



Ordinals
$P[Ordinal]$

$$(\forall \alpha. \qquad \rightarrow P[\alpha])$$
$$\rightarrow$$
$$\forall \alpha . P[\alpha]$$

Transfinite Induction

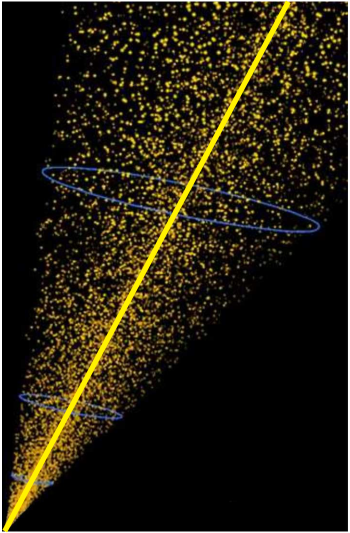


Ordinals

$P[Ordinal]$

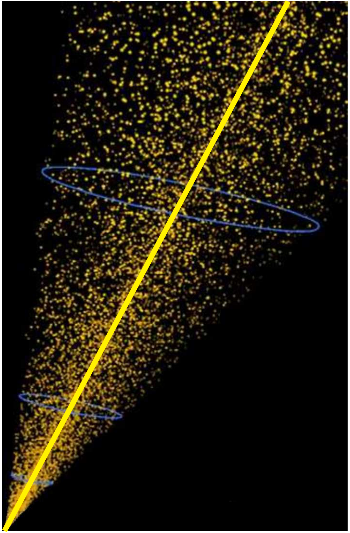
$$(\forall\alpha.(\forall\beta \in \alpha \,.\, P[\beta]) \to P[\alpha])$$
$$\to$$
$$\forall\alpha.P[\alpha]$$

Transfinite Induction and ∈-Induction



$P[set]$

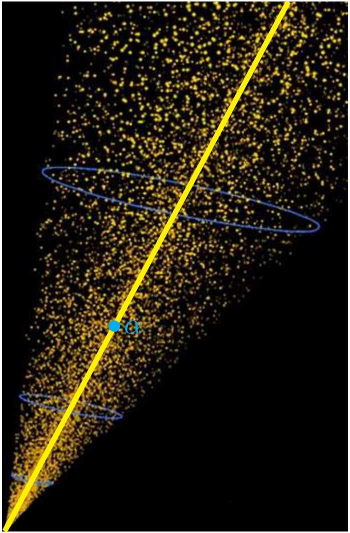
Transfinite Induction and $\in$-Induction



$$P[set]$$

Let $Q[\alpha]$ be $\forall Y. Y \subseteq V_\alpha \to P[Y]$.

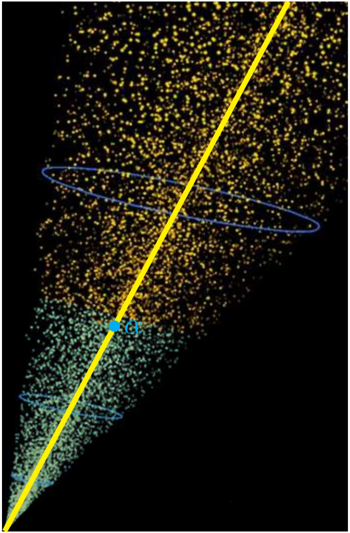
Transfinite Induction and $\in$-Induction



$$P[set]$$

Let $Q[\alpha]$ be $\forall Y. Y \subseteq V_\alpha \rightarrow P[Y]$.

Transfinite Induction and $\in$-Induction



$$P[set]$$

Let $Q[\alpha]$ be $\forall Y. Y \subseteq V_\alpha \to P[Y]$.

Transfinite Induction and $\in$-Induction

- To prove $\in$-induction we need:
- Transfinite Induction and two facts:

Transfinite Induction and $\in$-Induction

- To prove $\in$-induction we need:
- Transfinite Induction and two facts:
  1. If $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.

Transfinite Induction and $\in$-Induction

- To prove $\in$-induction we need:
- Transfinite Induction and two facts:
  1. If $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
  2. For every set Y, there is an α such that $Y \subseteq V_\alpha$.

Transfinite Induction and $\in$-Induction

- To prove $\in$-induction we need:
- Transfinite Induction and two facts:
  1. If $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
  2. For every set Y, there is an α such that $Y \subseteq V_\alpha$.
- Does the MML include V_α? (Yes...)

Transfinite Induction and $\in$-Induction

- To prove $\in$-induction we need:
- Transfinite Induction and two facts:
  1. If $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
  2. For every set Y, there is an α such that $Y \subseteq V_\alpha$.
- Does the MML include V_α? (Yes...)
- Search for theorems of the form $\forall Y.\exists \alpha. Y \subseteq *(\alpha)$.

Searching the MML for V

```
include('allofmizar-preface-types-axioms').

thf(invset,conjecture,
       (? [V: ($i>$i)] :
          (! [Y:$i] : (? [A:$i] : ((v3_ordinal1 @ A) & (r1_tarski @ Y @ (V @ A))))))).
```

Searching the MML for V

```
include('allofmizar-preface-types-axioms').

thf(invset,conjecture,
       (? [V: ($i>$i)] :
          (! [Y:$i] : (? [A:$i] : ((v3_ordinal1 @ A) & (r1_tarski @ Y @ (V @ A))))))).
```

```
~/satallax/satallax.native -similar rankprop2
```

Searching the MML for V

```
Read it, initial_branch has 135002 elts
Searching for similar formulas.
Conjunctions (2)
imp (v3_ordinal1 *3) False
r1_tarski __2 (*1 *3)
Number of conclusion variants: 5
Total partial matches: 96
t62_classes1 similarity measure 0
Total of 1 of measure 0
eps_ax similarity measure 2
Total of 1 of measure 2
t11_ordinal3 similarity measure 3
t4_ordinal3 similarity measure 3
s2_finset_1 similarity measure 3
s17_ordinal2 similarity measure 3
Total of 4 of measure 3
s1_card_1 similarity measure 4
```

96 Partial Matches

Searching the MML for V

```
Read it, initial_branch has 135002 elts
Searching for similar formulas.
Conjunctions (2)
imp (v3_ordinal1 *3) False
r1_tarski __2 (*1 *3)
Number of conclusion variants: 5
Total partial matches: 96
t62_classes1 similarity measure 0
Total of 1 of measure 0
eps_ax similarity measure 2
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t11_ordinal3 similarity measure 3
t4_ordinal3 similarity measure 3
s2_finset_1 similarity measure 3
s17_ordinal2 similarity measure 3
Total of 4 of measure 3
s1_card_1 similarity measure 4
```

96 Partial Matches
Closest: CLASSES1:62

Searching the MML for V

```
Read it, initial_branch has 135002 elts
Searching for similar formulas.
Conjunctions (2)
imp (v3_ordinal1 *3) False
r1_tarski __2 (*1 *3)
Number of conclusion variants: 5
Total partial matches: 96
t62_classes1 similarity measure 0
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t4_ordinal3 similarity measure 3
s2_finset_1 similarity measure 3
s17_ordinal2 similarity measure 3
Total of 4 of measure 3
s1_card_1 similarity measure 4
```

96 Partial Matches

Closest: CLASSES1:62

```
theorem :: CLASSES1:62
  ex A st X c= Rank A;
```

Searching the MML for V

```
Read it, initial_branch has 135002 elts
Searching for similar formulas.
Conjunctions (2)
imp (v3_ordinal1 *3) False
r1_tarski __2 (*1 *3)
Number of conclusion variants: 5
Total partial matches: 96
t62_classes1 similarity measure 0
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s2_finset_1 similarity measure 3
s17_ordinal2 similarity measure 3
Total of 4 of measure 3
s1_card_1 similarity measure 4
```

96 Partial Matches
Closest: CLASSES1:62

```
theorem :: CLASSES1:62
  ex A st X c= Rank A;
```

Rank A corresponds to V_α.

Importing

Global
(Theorems, Definitions)
Library

Developments
(Documents/Articles/Theories)
Local

Importing

```
environ

 vocabularies TARSKI, ORDINAL1, CLASSES1;
 notations TARSKI, ORDINAL1, CLASSES1;
 constructors CLASSES1;
 theorems CLASSES1;
 schemes ORDINAL1;
```

Global
(Theorems, Definitions)
Library

import

Developments
(Documents/Articles/Theories)
Local

Importing

```
environ

  vocabularies TARSKI, ORDINAL1, CLASSES1;
  notations TARSKI, ORDINAL1, CLASSES1;
  constructors CLASSES1;
  theorems CLASSES1;
  schemes ORDINAL1;
```

Global
(Theorems, Definitions)
Library

import

Developments
(Documents/Articles/Theories)
Local

TARSKI: $\subseteq$
ORDINAL1: Ordinals, Transfinite Induction Scheme
CLASSES1: Rank, $\forall Y.\exists\alpha.Y \subseteq V_\alpha$

Partial Proof

scheme

$$\forall Y . P[Y]$$

$$\forall Y . (\forall x \in Y . P[x]) \to P[Y]$$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
```

$\forall Y . P[Y]$

$$\forall Y . (\forall x \in Y . P[x]) \to P[Y]$$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
```

Definition: $Q[\alpha] := \forall Y. Y \subseteq V_\alpha \to P[Y]$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
```

Definition: $Q[\alpha] := \forall Y. Y \subseteq V_\alpha \rightarrow P[Y]$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
```

$$\forall \alpha \,.\, Q[\alpha]$$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
```

$$\forall \alpha \, . \, Q[\alpha]$$

```
  A3: for A holds Q[A]
```

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];




  A3: for A holds Q[A]
  let Y;
```

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
```

```
  A3: for A holds Q[A]
  let Y;
```

$\exists \alpha . Y \subseteq V_\alpha$ by "Fact 2"

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];




  A3: for A holds Q[A]
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
```

$\exists\alpha.Y \subseteq V_\alpha$ by "Fact 2"

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
```

```
  A3: for A holds Q[A]
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
```

$P[Y]$ since $Q[\alpha]$ and $Y \subseteq V_\alpha$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];



  A3: for A holds Q[A]
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

$P[Y]$ since $Q[\alpha]$ and $Y \subseteq V_\alpha$

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];




  A3: for A holds Q[A]
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

(by Transfinite Induction)

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
```

A2: $\forall\alpha\,.\,(\forall\beta\in\alpha\,.\,Q[\beta])\rightarrow Q[\alpha]$

```
  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
  A2: for A st for B st B in A holds Q[B] holds Q[A]
  proof
      :: something to prove
  end;
  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
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  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
  A2: for A st for B st B in A holds Q[B] holds Q[A]
  proof
      :: something to prove
  end;
  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

- Remaining subproof: By A1 and "Fact 1."

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
  A2: for A st for B st B in A holds Q[B] holds Q[A]
  proof
      :: something to prove
  end;
  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

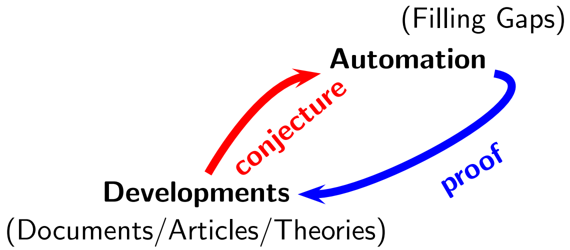
- Remaining subproof: By A1 and "Fact 1."
- Fact 1 can be proven as a lemma by another transfinite induction.

Partial Proof

```
scheme
  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
proof
  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
  A2: for A st for B st B in A holds Q[B] holds Q[A]
  proof
      :: something to prove
  end;
  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

- Remaining subproof: By A1 and "Fact 1."
- Fact 1 can be proven as a lemma by another transfinite induction.
- Given Fact 1, the gap can be completed by local reasoning. Automation?

Automation


(Filling Gaps)
Automation
conjecture
proof
Developments
(Documents/Articles/Theories)

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y \,.\, P[x]) \to P[Y]$
- Definition: $Q[\alpha] := \forall Y.Y \subseteq V_\alpha \to P[Y]$
- Goal: $\forall \alpha.(\forall \beta.\beta \in \alpha \to Q[\beta]) \to Q[\alpha]$

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
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- MPTP interface can try to automatically prove goals:

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
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- MPTP interface can try to automatically prove goals:

```
A2: for A st for B st B in A holds Q[B] holds Q[A] by;
```

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y . P[x]) \to P[Y]$
- Definition: $Q[\alpha] := \forall Y.Y \subseteq V_\alpha \to P[Y]$
- Goal: $\forall \alpha.(\forall \beta.\beta \in \alpha \to Q[\beta]) \to Q[\alpha]$
- MPTP interface can try to automatically prove goals:

```
A2: for A st for B st B in A holds Q[B] holds Q[A] ; :: ATP asked ...
```

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y . P[x]) \to P[Y]$
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- MPTP interface can try to automatically prove goals:

```
A2: for A st for B st B in A holds Q[B] holds Q[A] by EPSIND3E:1,A1,A2;
```

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y \,.\, P[x]) \to P[Y]$
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- Goal: $\forall \alpha.(\forall \beta.\beta \in \alpha \to Q[\beta]) \to Q[\alpha]$
- MPTP interface can try to automatically prove goals:

```
A2: for A st for B st B in A holds Q[B] holds Q[A] by EPSIND3E:1,A1,A2;
```

```
  A2: for A st for B st B in A holds Q[B] holds Q[A] by Th1,A1;
::>                                                       *4
```

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
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- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y . P[x]) \rightarrow P[Y]$
- Definition: $Q[\alpha] := \forall Y . Y \subseteq V_\alpha \rightarrow P[Y]$
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- Satallax can prove the corresponding HO THF problem in $< 0.1s$.

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
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- Goal: $\forall \alpha.(\forall \beta.\beta \in \alpha \rightarrow Q[\beta]) \rightarrow Q[\alpha]$
- Satallax can prove the corresponding HO THF problem in $< 0.1s$.
- Satallax can produce various kinds of proofs.
  - Coq scripts (using tactics to simulate tableau rules)

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y\,.\,P[x]) \to P[Y]$
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- Satallax can prove the corresponding HO THF problem in $< 0.1s$.
- Satallax can produce various kinds of proofs.
  - Coq scripts (using tactics to simulate tableau rules)
  - Coq proof terms (using lemmas corresponding to tableau rules)

Filling the Gap

- Lemma: For Y and α, if $Y \subseteq V_\alpha$ and $x \in Y$, then there is some $\beta \in \alpha$ such that $x \subseteq V_\beta$.
- Assumption: $\forall Y.(\forall x \in Y . P[x]) \to P[Y]$
- Definition: $Q[\alpha] := \forall Y.Y \subseteq V_\alpha \to P[Y]$
- Goal: $\forall \alpha.(\forall \beta.\beta \in \alpha \to Q[\beta]) \to Q[\alpha]$
- Satallax can prove the corresponding HO THF problem in $< 0.1s$.
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- Satallax can produce various kinds of proofs.
  - Coq scripts (using tactics to simulate tableau rules)
  - Coq proof terms (using lemmas corresponding to tableau rules)
  - But not Mizar proofs (yet?)
  - Mizar could be extended to allow "byproofterm" justifications with a small Curry-Howard-de Bruijn style proof checker.

Filling the Gap (Pf Term)

Satallax Produced Proof Term for the Gap:

```
exact (NNPP (forall (X1:set), v3_ordinal1 X1 -> (forall (X2:set), v3_ordinal1 X2 -> r2_hidden X2 X1 -> (forall (X3:set), r1_tars
ki X3 (k4_classes1 X2) -> p X3)) -> (forall (X2:set), r1_tarski X2 (k4_classes1 X1) -> p X2)) (fun H0 => TNAll set (fun (X1:set)
 => v3_ordinal1 X1 -> (forall (X2:set), v3_ordinal1 X2 -> r2_hidden X2 X1 -> q X2) -> q X1) H0 (fun __0 H1 => TNImp (v3_ordinal1
 __0) ((forall (X1:set), v3_ordinal1 X1 -> r2_hidden X1 __0 -> q X1) -> q __0) H1 (fun H2 H3 => TNImp (forall (X1:set), v3_ordin
al1 X1 -> r2_hidden X1 __0 -> q X1) (q __0) H3 (fun H4 H5 => TNAll set (fun (X1:set) => r1_tarski X1 (k4_classes1 __0) -> p X1)
H5 (fun __1 H6 => TNImp (r1_tarski __1 (k4_classes1 __0)) (p __1) H6 (fun H7 H8 => TAll set (fun (X1:set) => forall (X2:set), v3
_ordinal1 X2 -> r1_tarski X1 (k4_classes1 X2) -> (forall (X3:set), r2_hidden X3 X1 -> (exists X4:set, v3_ordinal1 X4 /\ r2_hidde
n X4 X2 /\ r1_tarski X3 (k4_classes1 X4)))) th1 __1 (fun H9 => TAll set (fun (X1:set) => v3_ordinal1 X1 -> r1_tarski __1 (k4_cla
sses1 X1) -> (forall (X2:set), r2_hidden X2 __1 -> (exists X3:set, v3_ordinal1 X3 /\ r2_hidden X3 X1 /\ r1_tarski X2 (k4_classes
1 X3)))) H9 __0 (fun H10 => TAll set (fun (X1:set) => (forall (X2:set), r2_hidden X2 X1 -> p X2) -> p X1) a1 __1 (fun H11 => TIm
p (v3_ordinal1 __0) (r1_tarski __1 (k4_classes1 __0) -> (forall (X1:set), r2_hidden X1 __1 -> (exists X2:set, v3_ordinal1 X2 /\
r2_hidden X2 __0 /\ r1_tarski X1 (k4_classes1 X2)))) H10 (fun H12 => H12 H2) (fun H12 => TImp (r1_tarski __1 (k4_classes1 __0))
(forall (X1:set), r2_hidden X1 __1 -> (exists X2:set, v3_ordinal1 X2 /\ r2_hidden X2 __0 /\ r1_tarski X1 (k4_classes1 X2))) H12
(fun H13 => H13 H7) (fun H13 => TImp (forall (X1:set), r2_hidden X1 __1 -> p X1) (p __1) H11 (fun H14 => TNAll set (fun (X1:set)
 => r2_hidden X1 __1 -> p X1) H14 (fun __4 H15 => TNImp (r2_hidden __4 __1) (p __4) H15 (fun H16 H17 => TAll set (fun (X1:set) =
> r2_hidden X1 __1 -> (exists X2:set, v3_ordinal1 X2 /\ r2_hidden X2 __0 /\ r1_tarski X1 (k4_classes1 X2))) H13 __4 (fun H18 =>
TImp (r2_hidden __4 __1) (exists X1:set, v3_ordinal1 X1 /\ r2_hidden X1 __0 /\ r1_tarski __4 (k4_classes1 X1)) H18 (fun H19 => H
19 H16) (fun H19 => TEx set (fun (X1:set) => v3_ordinal1 X1 /\ r2_hidden X1 __0 /\ r1_tarski __4 (k4_classes1 X1)) H19 (fun __5
H20 => TAnd (v3_ordinal1 __5) (r2_hidden __5 __0 /\ r1_tarski __4 (k4_classes1 __5)) H20 (fun H21 H22 => TAnd (r2_hidden __5 __0
) (r1_tarski __4 (k4_classes1 __5)) H22 (fun H23 H24 => TAll set (fun (X1:set) => v3_ordinal1 X1 -> r2_hidden X1 __0 -> q X1) H4
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```

If Mizar Accepted Proof Terms...

```
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  EpsilonInduction { P[set] } : for Y holds P[Y]
provided
A1: for Y st for x st x in Y holds P[x] holds P[Y]
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  defpred Q[Ordinal] means for Y st Y c= Rank $1 holds P[Y];
  A2: for A st for B st B in A holds Q[B] holds Q[A]
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 => v3_ordinal1 X1 -> (forall (X2:set), v3_ordinal1 X2 -> r2_hidden X2 X1 -> q X2) -> q X1) H0 (fun __0 H1 => TNImp (v3_ordinal1
 __0) ((forall (X1:set), v3_ordinal1 X1 -> r2_hidden X1 __0 -> q X1) -> q __0) H1 (fun H2 H3 => TNImp (forall (X1:set), v3_ordin
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```
        ;
  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```

If Mizar Accepted Proof Terms...

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 consider A such that A4: Y c= Rank A by CLASSES1:62;
 thus P[Y] by A3,A4;
end;
```

- Problem: Not Robust.

Hand-Written Mizar Subproof

```
A2: for A st for B st B in A holds Q[B] holds Q[A]
proof
  let A be Ordinal;
  assume B1: for B st B in A holds Q[B];
  let Y;
  assume B2: Y c= Rank A;
  for x st x in Y holds P[x]
  proof
    let x;
    assume x in Y;
    then consider B such that B3: B in A & x c= Rank B by Th1, B2;
    thus P[x] by B1,B3;
  end;
  hence P[Y] by A1;
end;
```

Hand-Written Mizar Subproof

```
tab_start H0.
tab_negall H0 __0 H1.
tab_negimp H1 H2 H3.
tab_negimp H3 H4 H5.
tab_negall H5 __1 H6.
tab_negimp H6 H7 H8.
tab_all th1 (__1) H9.
tab_all H9 (__0) H10.
tab_all a1 (__1) H11.
tab_imp H10 H12.
 tab_conflict H2 H12.
 tab_imp H12 H13.
  tab_conflict H7 H13.
  tab_imp H11 H14.
   tab_negall H14 __4 H15.
   tab_negimp H15 H16 H17.
   tab_all H13 (__4) H18.
   tab_imp H18 H19.
    tab_conflict H16 H19.
    tab_ex H19 __5 H20.
    tab_and H20 H21 H22.
    tab_and H22 H23 H24.
    tab_all H4 (__5) H25.
    tab_imp H25 H26.
     tab_conflict H21 H26.
     tab_imp H26 H27.
      tab_conflict H23 H27.
      tab_all H27 (__4) H28.
      tab_imp H28 H29.
       tab_conflict H24 H29.
       tab_conflict H29 H17.
   tab_conflict H14 H8.
```

```
A2: for A st for B st B in A holds Q[B] holds Q[A]
proof
  let A be Ordinal;
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  let Y;
  assume B2: Y c= Rank A;
  for x st x in Y holds P[x]
  proof
    let x;
    assume x in Y;
    then consider B such that B3: B in A & x c= Rank B by Th1, B2;
    thus P[x] by B1,B3;
  end;
  hence P[Y] by A1;
end;
```

Hand-Written Mizar Subproof

```
tab_start negA2.
tab_negall negA2 A H1.
tab_negimp H1 Aord H3.
tab_negimp H3 B1 H5.
tab_negall H5 Y H6.
tab_negimp H6 B2 negPY.
tab_all Th1 (Y) H9.
tab_all H9 (A) H10.
tab_all A1 (Y) H11.
tab_imp H10 H12.
 tab_conflict Aord H12.
 tab_imp H12 H13.
  tab_conflict B2 H13.
  tab_imp H11 H14.
   tab_negall H14 x H15.
   tab_negimp H15 x_in_Y negPx.
   tab_all H13 (x) H18.
   tab_imp H18 H19.
    tab_conflict x_in_Y H19.
    tab_ex H19 B H20.
    tab_and H20 Bord B3.
    tab_and B3 H23 H24.
    tab_all B1 (B) H25.
    tab_imp H25 H26.
     tab_conflict Bord H26.
     tab_imp H26 H27.
      tab_conflict H23 H27.
      tab_all H27 (x) H28.
      tab_imp H28 H29.
       tab_conflict H24 H29.
       tab_conflict H29 negPx.
   tab_conflict H14 negPY.
```

```
A2: for A st for B st B in A holds Q[B] holds Q[A]
proof
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   proof
      let x;
      assume x in Y;
      then consider B such that B3: B in A & x c= Rank B by Th1, B2;
      thus P[x] by B1,B3;
   end;
   hence P[Y] by A1;
end;
```

Proof of Epsilon Induction

```
scheme
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A1: for Y st for x st x in Y holds P[x] holds P[Y]
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    let Y;
    assume B2: Y c= Rank A;
    for x st x in Y holds P[x]
    proof
      let x;
      assume x in Y;
      then consider B such that B3: B in A & x c= Rank B by Th1, B2;
      thus P[x] by B1,B3;
    end;
    hence P[Y] by A1;
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  A3: for A holds Q[A] from ORDINAL1:sch 2(A2);
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      thus P[x] by B1,B3;
    end;
    hence P[Y] by A1;
  end;
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  let Y;
  consider A such that A4: Y c= Rank A by CLASSES1:62;
  thus P[Y] by A3,A4;
end;
```
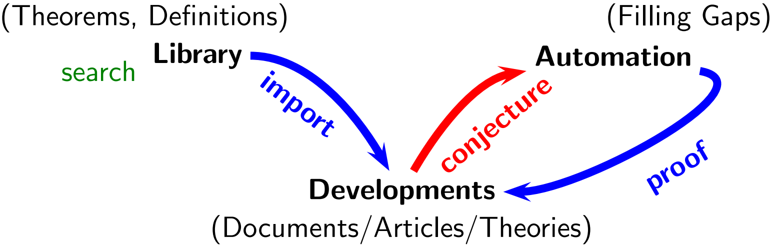
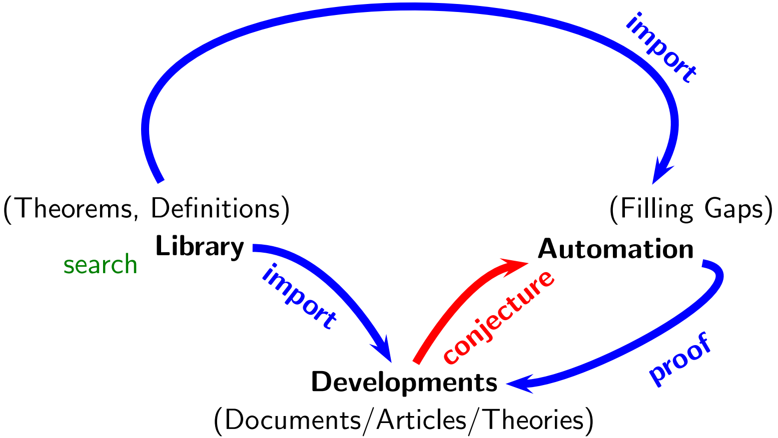

Actually, MPTP could prove this too, finding the reference.

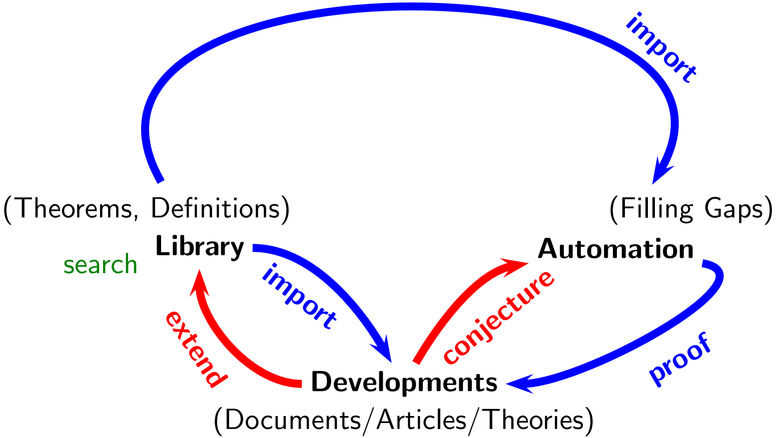
Three Legs Supporting the Formalization Process



Three Legs Supporting the Formalization Process



Three Legs Supporting the Formalization Process



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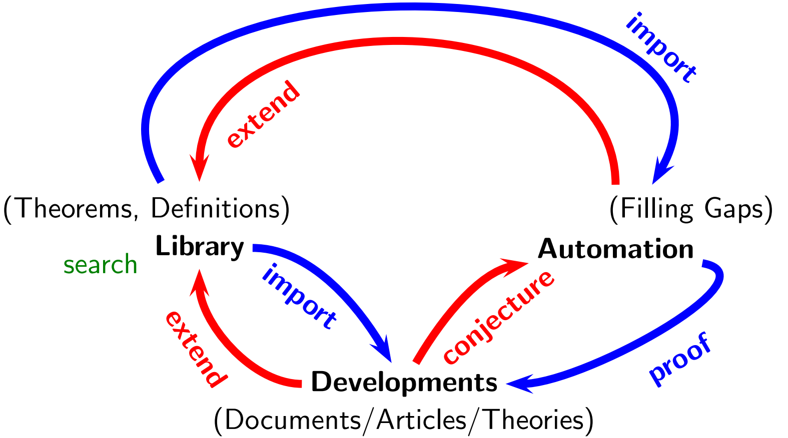

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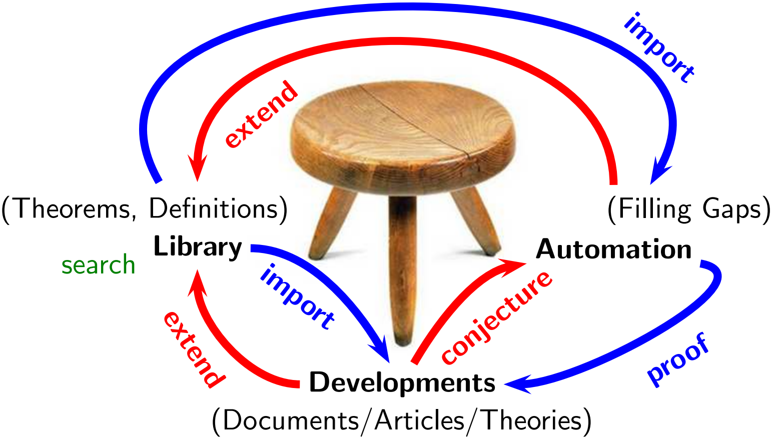

Outline

Ideal Mathematical Library

- Given a closed term t, library tells you a constant d defined by $d = t$.

Ideal Mathematical Library

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- Queries for exact matches (of definitions or theorems) can be immediate (hashing).
- Bad for searching, but fine for importing.

Library via Hashing: Knowns

- $\in$-induction:

$$\forall P.(\forall Y.(\forall x \in Y\,.\,P[x]) \to P[Y]) \to \forall Y.P[Y]$$

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scheme
  OtherEpsInduction { Q[set] } : for Z holds Q[Z]
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- Don't even need to know the article where it was proven.

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 let Y be set;
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- Opaque importation of definitions

Library via Hashing

- Represent the theorem $\forall Y. U(I(Y)) = Y$ as $\forall Y. \sharp e(\sharp d(Y))$.
- Serialize nameless rep of $\forall Y. \sharp e(\sharp d(Y))$ and hash.
- Remember hash is "known." In new developments this should be accepted:

```
theorem UIThm: for Y holds U(I(Y)) = Y by known;
```

Abstract Library

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Importing a development into the library increases $\mathcal{D}$ and/or $\mathcal{K}$.

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- Problems for having complete ATPs:
  - Polymorphism (type variables)
  - Type Definitions using Predicates

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- (Henkin) Complete ATPs for STT Exist

SATALLAX

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- Automation: More than FO *...but not too much more*